The Equational Theory of Algebras of Languages

BLAST in Nashville

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Outline

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Words and languages

Definitions

Notations

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$\{a, b\}^{\star}$ is the set of words over the alphabet $\{a, b\}$.

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$\{a, b\}^{+}$ is the set of non-empty words over the alphabet $\{a, b\}$.

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- The union and the intersection are written $\cup$ and $\cap$, and the empty language is 0.

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$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{(associativity of concatenation)}$$

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Language equivalence

$$\mathrm{Lang} \models e \simeq f \text{ iff } \forall\Sigma, \forall\sigma : X \to \mathcal{P}(\Sigma^\star), \widehat{\sigma}(e) = \widehat{\sigma}(f).$$

Free representation

$$r : \mathbb{E}_X \to R$$

$$r(e) = r(f)$$

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$$r(e) = r(f) \iff \mathrm{Lang} \models e \simeq f$$

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$r(e) = r(f)$ → Decidability

Outline

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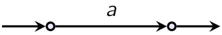
Idea

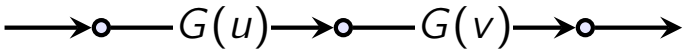
Compare 1-free terms under the assumption that certain variables contain ε.

Comparing series parallel terms

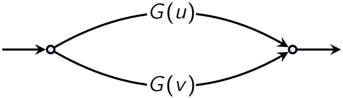
$\mathcal{G}(a) :=$



$\mathcal{G}(u \cdot v) :=$

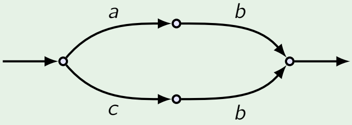


$\mathcal{G}(u \cap v) :=$

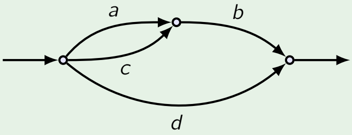


Example

$\mathcal{G}\left(((a \cap c) \cdot b) \cap d\right) =$



$\mathcal{G}\left((a \cdot b) \cap (c \cdot b)\right) =$

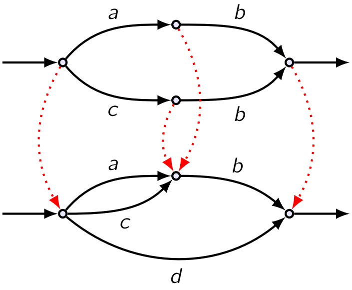


Preorder

Preorder on graphs

$G \lhd H$ if there exists a graph morphism from H to G.

H :



$(a \cdot b) \cap (c \cdot b)$

G :

$((a \cap c) \cdot b) \cap d$

Characterization theorem

$$u, v \in \mathbb{SP}_\Sigma ::= a \mid u \cdot v \mid u \cap v$$

Theorem

$$\mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \lhd \mathcal{G}(v)$$

Freyd & Scedrov, **Categories, Allegories**, *1990*

Andréka & Bredikhin, **The equational theory of union-free algebras of relations**, *1995*

Theorem

$$\forall u, v \in \mathbb{SP}_\Sigma,\ \mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathrm{Lang} \models u \subseteq v.$$

Andréka, Mikulás & Németi, **The equational theory of Kleene lattices**, *2011*

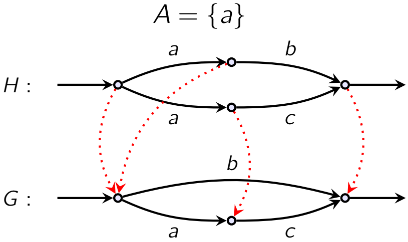
Preorder on weak graphs

Definition

A weak graph is a pair of a graph and a set of tests.

Weak graph preorder

$\langle G, A \rangle \blacktriangleleft \langle H, B \rangle$ if $B \subseteq A$ and there is an A-weak morphism from H to G.



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For every term $u \in \mathbb{T}_X$ we can build a weak graph $\mathcal{G}(u)$.

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For every term $u \in \mathbb{T}_X$ we can build a weak graph $\mathcal{G}(u)$.

Corollary

$$\text{Lang} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v).$$

Simplifying expressions

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$$\mathrm{Lang} \models e \subseteq f \Leftrightarrow \forall u \in \mathcal{T}\left(e\right), \exists v \in \mathcal{T}\left(f\right) : \mathrm{Lang} \models u \subseteq v$$

Free representation of expressions

$$\mathbb{E}_X$$

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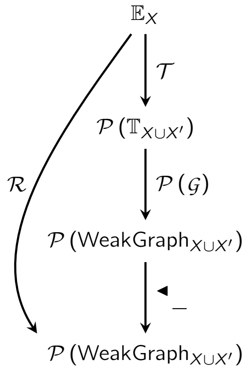

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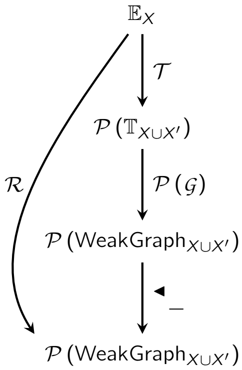

Theorem

$$\text{Lang} \models e \simeq f \Leftrightarrow \mathcal{R}(e) = \mathcal{R}(f)$$

Outline

Axiomatisation

For series parallel terms

Assume a set of tests A,

$$\frac{}{A \vdash_{sp} u \leq u} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} v \leq w}{A \vdash_{sp} u \leq w}$$

$$\frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cdot u' \leq v \cdot v'} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cap u' \leq v \cap v'}$$

$$\frac{}{A \vdash_{sp} u \cdot (v \cdot w) = (u \cdot v) \cdot w} \qquad \frac{}{A \vdash_{sp} u \cap (v \cap w) = (u \cap v) \cap w}$$

$$\frac{}{A \vdash_{sp} u \leq u \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq v \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq u}$$

$$\frac{var(u) \subseteq A}{A \vdash_{sp} v \leq u \cdot v} \qquad \frac{var(u) \subseteq A}{A \vdash_{sp} v \leq v \cdot u}$$

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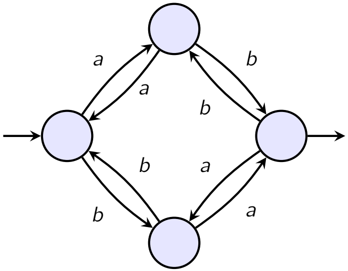
Decidability of Kleene algebra

Finite state automata

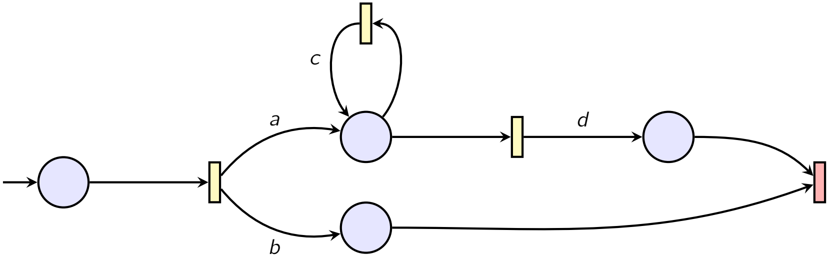
$$e, f ::= 0 \mid 1 \mid a \mid e \cup f \mid e \cdot f \mid e^{\star}.$$



Decidability of identity-free Kleene lattices

Petri automata

$$e, f ::= 0 \mid a \mid e \cup f \mid e \cap f \mid e \cdot f \mid e^+.$$



Decidability of Language algebra

Weighted Petri automata

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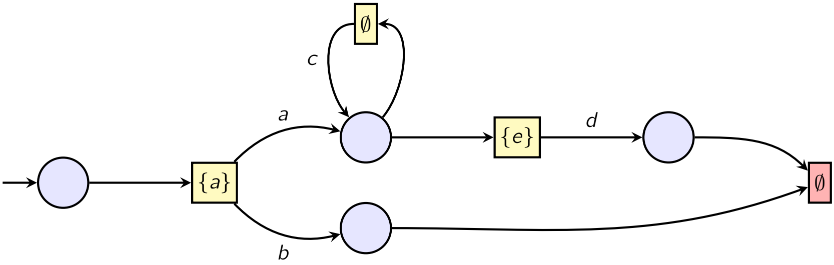

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Open problems

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   No finite axiomatisation, but maybe finitely presentable?

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   No finite axiomatisation, but maybe finitely presentable?
II. What about $\top$?

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Just add $e \subseteq \top$ and $\top \subseteq \top^\smile$, there is an equivalent graph construction.
However, completeness is tricky...

That's all folks!

Thank you!

Freyd & Scedrov, **Categories, Allegories**, *1990*

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See more at:

`http://paul.brunet-zamansky.fr`

Outline