A formal exploration of **Nominal Kleene Algebra**

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Introduction

$$x \leftarrow 1; choice(y \leftarrow x | y \leftarrow 0) \qquad choice(x \leftarrow 1; y \leftarrow x | x \leftarrow 1; y \leftarrow 0)$$

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$$\begin{array}{ccc} x \leftarrow 1; \mathit{choice}(y \leftarrow x | y \leftarrow 0) & \qquad & \mathit{choice}(x \leftarrow 1; y \leftarrow x | x \leftarrow 1; y \leftarrow 0) \\ \big\downarrow & & \big\downarrow \\ a \cdot (b + c) & & (a \cdot b) + (a \cdot c) \end{array}$$

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Fundamental Theorem of Kleene Algebra

$\forall e, f \in Reg\langle\Sigma\rangle$,

$$Rel \models e = f \quad \Leftrightarrow \quad KA \vdash e = f \quad \Leftrightarrow \quad L(e) = L(f)$$

Towards Nominal Kleene Algebra

A simple swap program

$$t \leftarrow x \qquad ; \qquad x \leftarrow y \qquad ; \qquad y \leftarrow t$$

Towards Nominal Kleene Algebra

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$$t \leftarrow x \quad ; \quad x \leftarrow y \quad ; \quad y \leftarrow t$$

$$tx \qquad xy \qquad yt$$

tx:

t	v_1
x	v_2
y	v_3

t	v_2
x	v_2
y	v_3

xy:

t	v_1
x	v_2
y	v_3

t	v_1
x	v_3
y	v_3

yt:

t	v_1
x	v_2
y	v_3

t	v_1
x	v_2
y	v_1

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$$(\nu t. \quad tx \quad \circ \quad xy \quad \circ \quad yt \quad)$$

x	v_2
y	v_3

t	v_1
x	v_2
y	v_3

t	v_2
x	v_3
y	v_2

x	v_3
y	v_2

Outline

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Nominal Kleene Algebra

Variations around Nominal Kleene Algebra

Results

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Nominal set

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$()\,,(a\ b)\,,\pi,\cdots \in \mathfrak{S}$ the set of finitely supported permutations over $\mathcal{A}$.

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X is a set and $\bowtie$ is a map $\mathfrak{S} \times X \to X$. X is a **nominal set** over $\mathcal{A}$ if:

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$a \# x$ means a is not in the support of x.

Nominal Kleene Algebra

$$\mathcal{K} = \langle K, 0, 1, +, \cdot, ^\star, \nu \rangle$$

---

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- $\mathcal{K}$ satisfies the **nominal axioms**(i):

$$\begin{aligned}
\nu a.\,(e+f) &= \nu a.e + \nu a.f & \qquad a \# e &\Rightarrow \nu b.e = \nu a.(a\ b) \bowtie e \\
\nu a.\,(\nu b.e) &= \nu b.\,(\nu a.e) & a \# e &\Rightarrow \nu a.f \cdot e = \nu a.\,(f \cdot e) \\
a \# e &\Rightarrow \nu a.e = e & a \# e &\Rightarrow e \cdot \nu a.f = \nu a.\,(e \cdot f)
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- $\mathcal{K}$ satisfies the **permutation axioms**:

$$\begin{array}{lll} \pi \bowtie 1 \coloneqq 1 & \pi \bowtie (\nu_a.e) \coloneqq \nu_{\pi(a)}.\,(\pi \bowtie e) & \pi \bowtie (e^\star) \coloneqq (\pi \bowtie e)^\star \\ \pi \bowtie 0 \coloneqq 0 & \pi \bowtie (e \cdot f) \coloneqq \pi \bowtie e \cdot \pi \bowtie f & \pi \bowtie (e+f) \coloneqq \pi \bowtie e + \pi \bowtie f \end{array}$$

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Completeness of NKA

$$e, f ::= 0 \mid 1 \mid a \mid e + f \mid e \cdot f \mid e^\star \mid \nu a.e$$

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$L(e) = L(f)$ is **decidable**.[iii]

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$\Rightarrow$ We need to change the syntax of expressions, and adapt the axiomatisation.

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Modified nominal expressions

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$$e, f ::= 0 \mid 1 \qquad \mid a \mid x \mid e{+}f \mid e{\cdot}f \mid e^{\star} \mid \nu a.e$$

$x, y, \dots \in X$ a nominal set over $\mathcal{A}$.

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- every element of X has a type $A \in \mathcal{P}_f(\mathcal{A})$.

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Questions:

1. What axiomatisations should we give for these expressions?
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3. Could we salvage some results of NKA?

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$$\frac{}{Ax \vdash e = f}$$

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- Structural rules:

$$\frac{Ax \vdash f = e}{Ax \vdash e = f} \qquad \frac{Ax \vdash e = f \quad Ax \vdash f = g}{Ax \vdash e = f} \qquad \frac{}{Ax \vdash e = f}\,(e, f) \in Ax$$

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- Congruence rules:

$$\frac{}{Ax \vdash 0 = 0} \qquad \frac{}{Ax \vdash id_A = id_A} \qquad \frac{Ax \vdash e = g \quad Ax \vdash f = h}{Ax \vdash e + f = g + h} \qquad \dots$$

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- Constant-free Kleene Algebra axioms:

$$\frac{}{Ax \vdash e + f = f + e} \qquad \frac{}{Ax \vdash e \cdot (f + g) = (e \cdot f) + (e \cdot g)}$$

$$\frac{Ax \vdash f + e \cdot g \leqslant g}{Ax \vdash e^\star \cdot f \leqslant g} \qquad \dots$$

Theories

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Ordering theories

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Embedding preorder

$(S, Ax) \preccurlyeq (S', Ax')$ if there is a function $\varphi : S \to S'$ such that $\forall e, f \in S$:

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Reduction preorder

$(S, Ax) \ll (S', Ax')$ if for every pair $\langle e, f \rangle \in S \times S$ there is a pair $\langle e', f' \rangle \in S' \times S'$ such that

$$Ax \vdash e = f \Leftrightarrow Ax' \vdash e' = f'.$$

Outline

Results

NKA

$\longrightarrow$: embedding

$\dashrightarrow$: reduction

Results

$$\begin{array}{c} \text{NKA} \\ \downarrow \\ \mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U} \cap \mathbb{C} \end{array}$$

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Results

NKA

$$\mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U} \cap \mathbb{C} \leftrightarrows \mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{U} \cap \mathbb{C}$$

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Results

NKA

$$\begin{tikzcd} \mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U} \arrow[r, dashed, shift right] & \mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{U} \arrow[l, shift right] \end{tikzcd}$$

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$$
\begin{tikzcd}
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\end{tikzcd}
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Results

$$
\begin{tikzcd}
\textcolor{red}{\text{NKA}} \arrow[d, red] & \mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U} \arrow[r, dashed, shift right] & \mathbb{E}^{+}\langle\mathcal{A}\rangle \cap \mathbb{U} \arrow[l, shift right] \arrow[dl, leftrightarrow] \\
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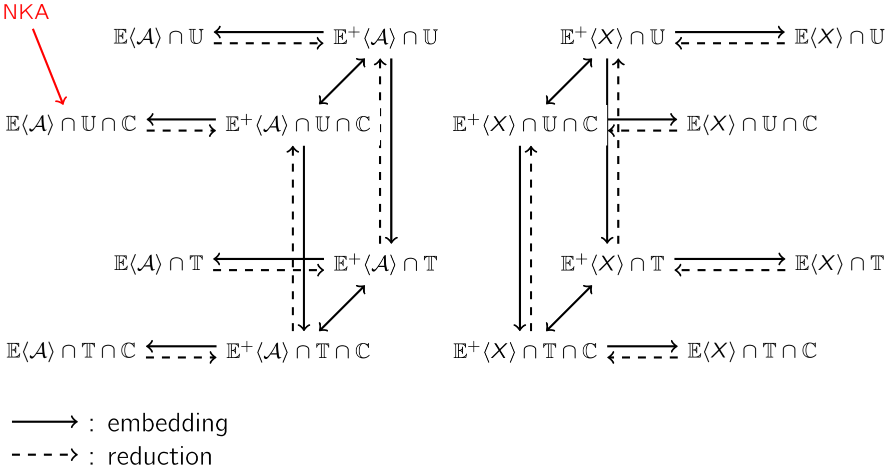
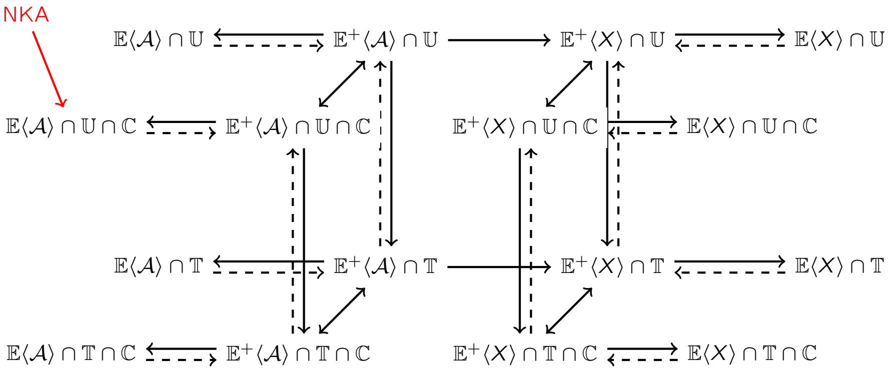
Results


NKA
$\mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U}$
$\mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{U}$
$\mathbb{E}^+\langle X\rangle \cap \mathbb{U}$
$\mathbb{E}\langle X\rangle \cap \mathbb{U}$
$\mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{U} \cap \mathbb{C}$
$\mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{U} \cap \mathbb{C}$
$\mathbb{E}^+\langle X\rangle \cap \mathbb{U} \cap \mathbb{C}$
$\mathbb{E}\langle X\rangle \cap \mathbb{U} \cap \mathbb{C}$
$\mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{T}$
$\mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{T}$
$\mathbb{E}^+\langle X\rangle \cap \mathbb{T}$
$\mathbb{E}\langle X\rangle \cap \mathbb{T}$
$\mathbb{E}\langle\mathcal{A}\rangle \cap \mathbb{T} \cap \mathbb{C}$
$\mathbb{E}^+\langle\mathcal{A}\rangle \cap \mathbb{T} \cap \mathbb{C}$
$\mathbb{E}^+\langle X\rangle \cap \mathbb{T} \cap \mathbb{C}$
$\mathbb{E}\langle X\rangle \cap \mathbb{T} \cap \mathbb{C}$
: embedding
: reduction

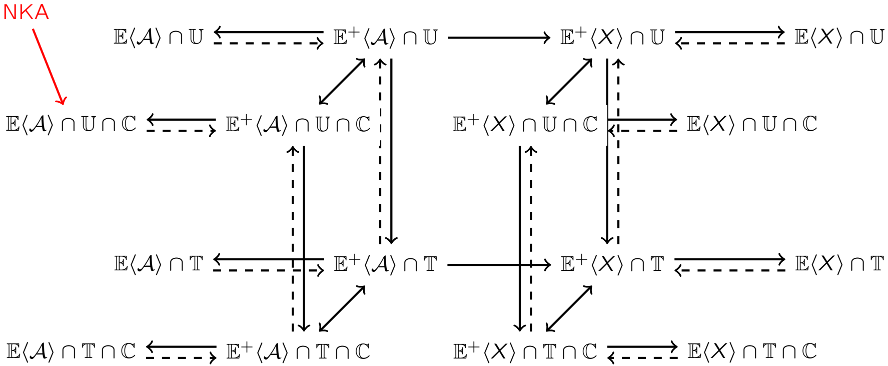
Results



$\longrightarrow$: embedding

$\dashrightarrow$: reduction

Results

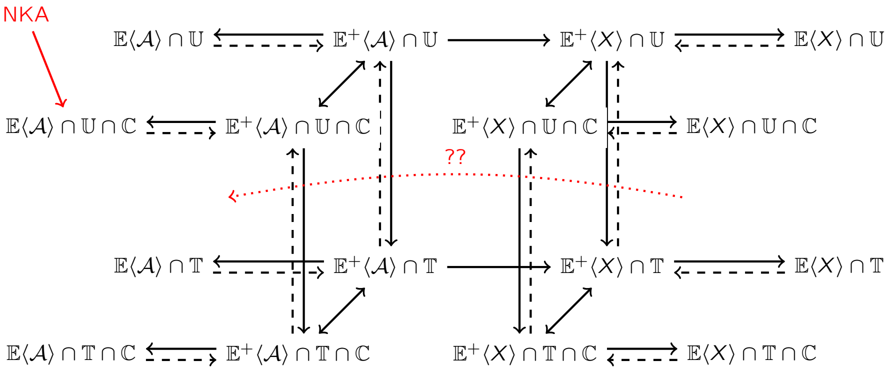


$\longrightarrow$: embedding

$\dashrightarrow$: reduction

Proved using the proof assistant Coq ($\sim$ 9000 lines of proof script).
`http://perso.ens-lyon.fr/paul.brunet/nka`

Results



$\longrightarrow$: embedding

$\dashrightarrow$: reduction

Proved using the proof assistant Coq ($\sim$ 9000 lines of proof script).

`http://perso.ens-lyon.fr/paul.brunet/nka`

That's all folks!

Thank you!

See more at:
`http://perso.ens-lyon.fr/paul.brunet`

Outline

Introduction

Nominal Kleene Algebra

Variations around Nominal Kleene Algebra

Results