Equivalence of regular expressions with converse on relations

An alternative presentation of the proof by **Bloom**, **Ésik** and **Stefanescu**

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Converse I

Converse on languages : Mirror

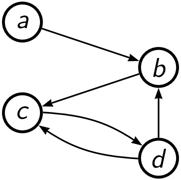
$$1^\vee := 1$$

$$(x \cdot w)^\vee := w^\vee \cdot x$$

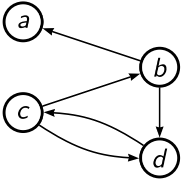
$$L^\vee := \{w^\vee \mid w \in L\}$$

Converse on relations

$$R^\vee := \{(y, x) \mid (x, y) \in R\}$$



Converse II



Plan

1. Kleene Algebrae with converse
2. Construction of the closure of an automaton
3. On examples

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Equivalence

We'll use different notions of equivalence on expressions e, f on an alphabet X :

Language equality : $[\![e]\!] = [\![f]\!]$;
Equivalence for language models : $\forall \Sigma, \forall \sigma \in {\Sigma^*}^X, \hat{\sigma}(e) = \hat{\sigma}(f)$: $\boxed{e \equiv_{Lang} f}$;
Equivalence for relation models : $\forall S, \forall \sigma \in \mathscr{P}\left(S^2\right)^X, \hat{\sigma}(e) = \hat{\sigma}(f)$: $\boxed{e \equiv_{Rel} f}$.

We write $[\![e]\!]$ for the language denoted by a regular expression e.

Kleene Algebrae I

A Kleene Algebra [(i)] is an algebraic structure $\langle K, +, \cdot, ^*, 0, 1\rangle$ satifying :

1. $\langle K, +, \cdot, 0, 1\rangle$ is an idempotent semiring :

$\langle K, +, 0\rangle$ is a commutative idempotent monoid	$a + (b + c) = (a + b) + c$ $a + b = b + a$ $a + 0 = a$ $a + a = a$
$\langle K, \cdot, 1\rangle$ is a monoid	$a(bc) = (ab)c$ $1a = a$ $a1 = a$
Distributivity laws	$a(b + c) = ab + ac$ $(a + b)c = ac + bc$ $0a = 0$ $a0 = 0$

Kleene Algebrae II

2. The * operation satisfy :

$$
\begin{aligned}
1 + aa^* &\leqslant a^* \\
1 + a^*a &\leqslant a^* \\
b + ax \leqslant x &\Rightarrow a^*b \leqslant x \\
b + xa \leqslant x &\Rightarrow ba^* \leqslant x
\end{aligned}
$$

Where $a \leqslant b \overset{\Delta}{\Leftrightarrow} a + b = b$.

The last axioms can be replaced by a number of things.

(i). As presented in [Koz94].

All is well

$$[\![e]\!] = [\![f]\!] \quad \Leftrightarrow \quad KA \vdash e = f \quad \Leftrightarrow \quad e \equiv_{Lang} f \quad \Leftrightarrow \quad e \equiv_{Rel} f$$

Kleene Algebra with Converse

Theorem [BES95]

A complete axiomatization of the variety $L^{\vee}$ generated by regular languages with converse consists of the axioms for KA and the following :

- $(a+b)^{\vee} = a^{\vee} + b^{\vee}$
- $(a \cdot b)^{\vee} = b^{\vee} \cdot a^{\vee}$
- $(a^*)^{\vee} = (a^{\vee})^*$
- $a^{\vee\vee} = a$.

Equivalence in $L^{\vee}$

Let $e, f \in Reg^{\vee}(X)$.
We compute $\tau(e), \tau(f) \in Reg(X \cup X')$.
Where

$$\begin{array}{l|l} \forall x \in X, \tau(x) = x & \forall x \in X, \nu(x) = x' \\ \tau(\mathbb{1}) = \mathbb{1} & \nu(\mathbb{1}) = \mathbb{1} \\ \tau(e_1 \cdot e_2) = \tau(e_1) \cdot \tau(e_2) & \nu(e_1 \cdot e_2) = \nu(e_2) \cdot \nu(e_1) \\ \tau(e_1 + e_2) = \tau(e_1) + \tau(e_2) & \nu(e_1 + e_2) = \nu(e_1) + \nu(e_2) \\ \tau(e^*) = \tau(e)^* & \nu(e^*) = \nu(e)^* \\ \tau(e^{\vee}) = \nu(e) & \nu(e^{\vee}) = \tau(e) \end{array}$$

and

$X' := \{x' \mid x \in X\}$ is a disjoint copy of X.
Furthermore

$$\boxed{[\![\tau(e)]\!] = [\![\tau(f)]\!] \Leftrightarrow e \equiv_{Lang} f}$$

Languages *vs.* Relations

$$a \leqslant aa^\vee a \text{ ???}$$

in $L^\vee$	in $REL^\vee$
$a \not\subseteq aa^\vee a = aaa$	if xRy then $xRyR^\vee xRy$
	which means $xRR^\vee Ry$
	so $R \subseteq RR^\vee R$

Theorem [EB95]

A complete set of axioms for the variety $REL^\vee$ generated by regular relations with converse consists on the axioms for $L^\vee$ and the axiom $a \leqslant aa^\vee a$.

Equivalence in $REL^{\vee}$

Theorem [BES95]

$$cl([\![\tau(e)]\!]) = cl([\![\tau(f)]\!]) \Leftrightarrow e \equiv_{Rel} f$$

Let $\bar{.}$ be the function $\begin{cases} (X \cup X')^* & \rightarrow & (X \cup X')^* \\ \epsilon & \mapsto & \epsilon \\ xw & \mapsto & \overline{w}x' \\ x'w & \mapsto & \overline{w}x \end{cases}$

We define the relation $\rightsquigarrow$ on $(X \cup X')^*$:

$$u \rightsquigarrow v \overset{\Delta}{\Leftrightarrow} \exists u_1, w, u_2 : u = u_1 w \overline{w} w u_2 \wedge v = u_1 w u_2.$$

$cl(A)$ is the closure of A for $\rightsquigarrow$: $cl(A) = \{v \mid \exists u \in A : u \rightsquigarrow^* v\}$.

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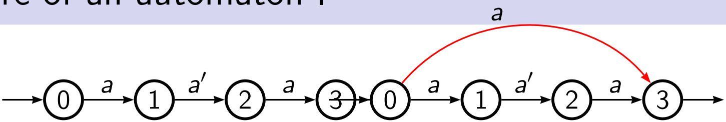
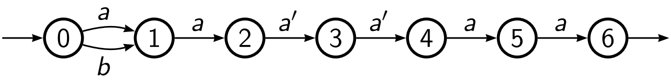
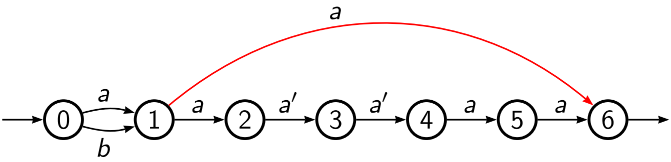
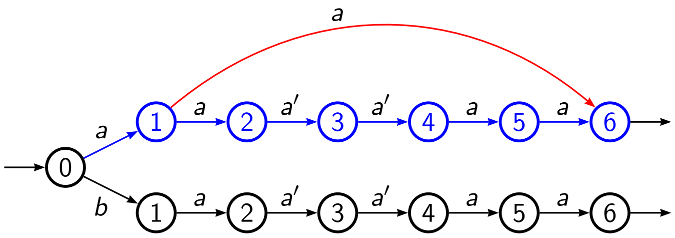
Closure of an automaton I

- 

- 



Closure of an automaton II



And it gets worse...

Idea of the construction

If $q_0 \xrightarrow{uw} q_1 \xrightarrow{x} q_2 \xrightarrow{\overline{wx}wx} q_3$,

i.e. pattern wx

then : $q_0, [\epsilon] \xrightarrow{uw} q_1, [uw] \xrightarrow{x} q_3, [uwx]$

Furthermore, if $q_0 \xrightarrow{uw} q_1 \xrightarrow{x} q_2 \xrightarrow{v} q_3$,

such that $v \Rightarrow^* x'\overline{w}wx$,

then : $q_0, [\epsilon] \xrightarrow{uw} q_1, [uw] \xrightarrow{x} q_3, [uwx]$

Construction : Γ

We write $\bar{X} := X \cup X'$.
For $w \in \bar{X}^*$, consider the language defined inductively :

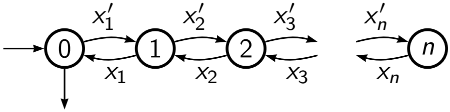
$$\begin{array}{rccc} \Gamma : & \bar{X}^* & \longrightarrow & \mathscr{P}(\bar{X}^*) \\ & \epsilon & \longmapsto & \epsilon \\ & wx & \longmapsto & (x'\Gamma(w)x)^* \end{array}$$

Properties of $\Gamma(w)$

1. $\Gamma(w)$ is *upward* $\rightsquigarrow$*-closed* : $u \rightsquigarrow v \in \Gamma(w) \Rightarrow u \in \Gamma(w)$;
2. $u \in \Gamma(w) \Rightarrow wu \rightsquigarrow^* w$;
3. $\exists v$ suffix of w : $u \rightsquigarrow^* \overline{v}v \Rightarrow u \in \Gamma(w)$;
4. subsequently : $\Gamma(w) = cl^{\uparrow}\left(\{\overline{v}v \mid v \text{ suffix of } w\}\right)$

$\Gamma(w)$ is upward-closed

$\Gamma(x_n \cdots x_1)$ is recognized by :



Property

In this automaton, if $q_1 \xrightarrow{x} q_2$, then $q_2 \xrightarrow{x'} q_1$.

Corollary

If $0 \xrightarrow{u_1} q_1 \xrightarrow{w} q_2 \xrightarrow{u_2} 0$, then

$$0 \xrightarrow{u_1} q_1 \xrightarrow{w} q_2 \xrightarrow{\overline{w}} q_1 \xrightarrow{w} q_2 \xrightarrow{u_2} 0 \;.$$

Construction : γ

Notations

We consider an automaton $\mathscr{A} = \langle Q, \bar{X}, I, F, \Delta \rangle$,
and define $\forall x \in \bar{X},\ R_x \coloneqq \{(q, q') \mid (q, x, q') \in \Delta\}$.

Definition : γ

$$\gamma : \begin{array}{ccc} \bar{X}^* & \longrightarrow & Rel_Q \\ \epsilon & \longmapsto & Id_Q \\ w \cdot x & \longmapsto & (R_{x'} \circ \gamma(w) \circ R_x)^* \end{array}$$

And $G = \bar{X}^*/\gamma = \{[w] \mid w \in \bar{X}^*\} = \{\{u \mid \gamma(u) = \gamma(w)\} \mid w \in \bar{X}^*\}$ (finite)

Property

$\gamma(w) = \hat{\sigma}(\Gamma(w))$ where $\sigma(x) = R_x$, which means

$$(q_1, q_2) \in \gamma(w) \Leftrightarrow (\exists u \in \Gamma(w) : q_1 \xrightarrow{u} q_2)$$

Closure Automaton

$cl(\mathscr{A})$

$$cl(\mathscr{A}) := \langle Q \times G, \bar{X}, I \times \{\mathbb{1}\}, F \times G, \Delta' \rangle$$

where $\Delta' := \{((q_1, [w]), x, (q_2, [wx])) \mid (q_1, q_2) \in R_x \circ \gamma(wx)\}$ Which is to say $(q_1, [w]) \xrightarrow{x} (q_2, [wx])$ if :

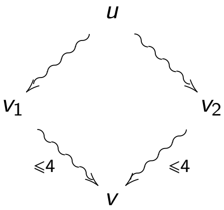
$$q_1 \xrightarrow{x} q_3 \text{ and } (q_3, q_2) \in \gamma(wx)$$

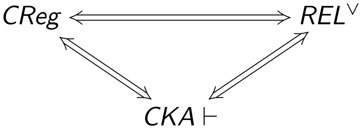
- The γ function needs to (or at least should) be pre-computed, but the rest of the contruction can be done *on-the-fly*.
- One can build a *deterministic* closure automaton with set of states $\mathscr{P}(Q) \times G$ and transition *function* :

$$\delta(\{q_1 \ldots q_n\}, [w]) := (\{p \mid \exists q_i : (q_i, p) \in R_x \circ \gamma(wx)\}, [wx]).$$

To conclude

- $\rightsquigarrow$ is confluent. Furthermore,



- 

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