Petri automata

for Kleene Allegories

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Regular expressions

$$e, f \in \mathrm{Reg}_X ::= 0 \mid 1 \mid x \in X \mid e \cdot f \mid e \cup f \mid e^\star$$

Interpretations

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- *languages:* Σ a finite set, $\sigma : X \to \mathcal{P}(\Sigma^\star)$,
  $\emptyset$, $\{\epsilon\}$, concatenation, union

Rationnal languages correspond to $[\![\_]\!] : \begin{array}{ccc} X & \to & \mathcal{P}(X^\star) \\ x & \mapsto & \{x\}. \end{array}$

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- *relations:* S a set, $\sigma : X \to \mathcal{P}(S \times S)$,
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Model equivalence

$e, f \in \mathrm{Reg}_X$

$$\mathrm{Rel} \models e = f \quad \text{if} \quad \forall S, \forall \sigma : X \to \mathcal{P}(S \times S), \sigma(e) = \sigma(f)$$

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Theorem

$$\mathrm{Rel} \models e = f \Leftrightarrow [\![e]\!] = [\![f]\!]$$

Kleene Allegories

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Kleene Allegories

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$$\mathrm{Rel} \models e = f \;\not\Leftarrow\; [\![e]\!] = [\![f]\!]$$

Example

$$[\![a \cap b]\!] = \emptyset = [\![0]\!]$$

$$\begin{aligned} \sigma(a) &= \{(x,y),(y,z)\} \\ \sigma(b) &= \{(y,z),(z,t)\} \\ \sigma(a \cap b) &= \{(y,z)\} \neq \emptyset = \sigma(0) \end{aligned}$$

$$[\![a]\!] = \{a\} = [\![a^\smile]\!]$$

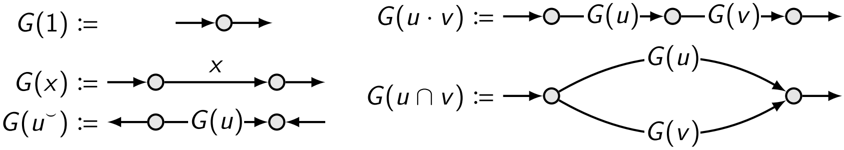
$$\begin{aligned} \sigma(a) &= \{(x,y)\} \\ \sigma(a^\smile) &= \{(y,x)\} \neq \sigma(a) \end{aligned}$$

A different approach is needed.

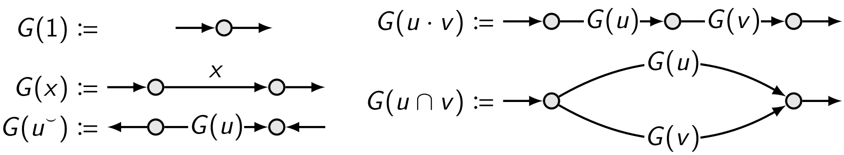
Graphs/Ground terms

$$u, v \in \mathsf{W}_X ::= 0 \mid 1 \mid x \in X \mid u \cdot v \mid u \cap v \mid u \cup v \mid u^\star \mid u^\smile$$

Graphs/Ground terms

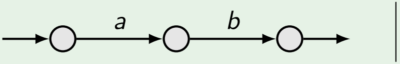


Graphs/Ground terms



Example

$G(a \cdot b)$:



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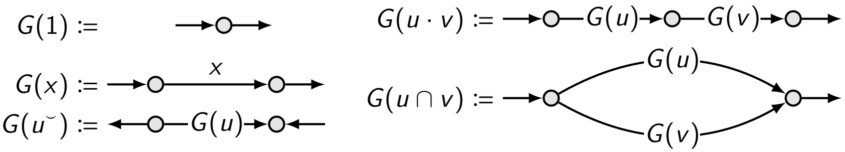
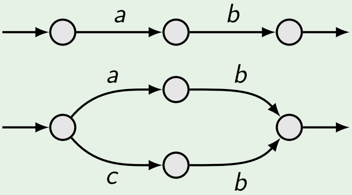


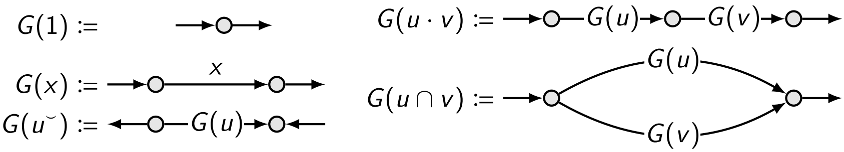
Example

$G\,(a \cdot b)$:

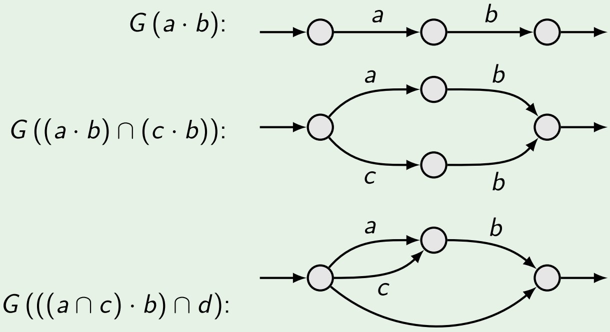
$G\,((a \cdot b) \cap (c \cdot b))$:



Graphs/Ground terms

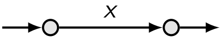


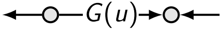
Example

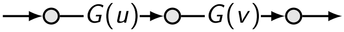


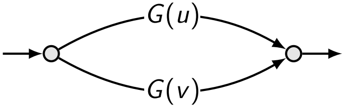
Graphs/Ground terms

$G(1) :=$ 

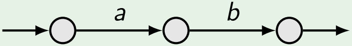
$G(x) :=$ 

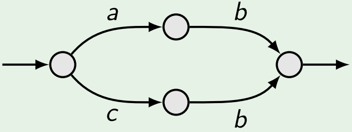
$G(u^{\smile}) :=$ 

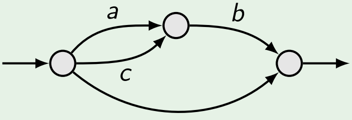
$G(u \cdot v) :=$ 

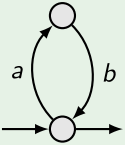
$G(u \cap v) :=$ 

Example

$G(a \cdot b)$: 

$G((a \cdot b) \cap (c \cdot b))$: 

$G(((a \cap c) \cdot b) \cap d)$: 

$G((a \cdot b) \cap 1)$: 

Characterization theorem

$\blacktriangleleft$ is a preorder on graphs.

Theorem

$u, v \in \mathsf{W}_X$,

$$\mathrm{Rel} \models u \leqslant v \Leftrightarrow G(u) \blacktriangleleft G(v)$$

- P. J. Freyd and A. Scedrov. *Categories, Allegories.* NH, 1990
- H. Andréka and D. Bredikhin. The equational theory of union-free algebras of relations. *Alg. Univ.*, 33(4):516–532, 1995

Graphs/Ground terms languages

$$[\![\_]\!] : \mathrm{Reg}_X^{\smile\cap} \to \mathcal{P}(\mathrm{W}_X)$$

$$\begin{aligned}
[\![0]\!] &:= \emptyset \\
[\![1]\!] &:= \{1\} \\
[\![x]\!] &:= \{x\} \\
[\![e^\smile]\!] &:= \{w^\smile \mid w \in [\![e]\!]\} \\
[\![e \cdot f]\!] &:= \{w \cdot w' \mid w \in [\![e]\!] \text{ and } w' \in [\![f]\!]\} \\
[\![e \cap f]\!] &:= \{w \cap w' \mid w \in [\![e]\!] \text{ and } w' \in [\![f]\!]\} \\
[\![e \cup f]\!] &:= [\![e]\!] \cup [\![f]\!] \\
[\![e^\star]\!] &:= \bigcup_{n\in\mathbb{N}} \{w_1 \cdot \dots \cdot w_n \mid \forall i, w_i \in [\![e]\!]\} \quad .
\end{aligned}$$

Graph language of an expression

$e \in \mathrm{Reg}_X^{\smile\cap}$,

$$G(e) := \{G(w) \mid w \in [\![e]\!]\} .$$

Characterization theorem

${}^{\blacktriangleleft}S$ is the downwards closure of S with respect to $\blacktriangleleft$.

Theorem

$e, f \in \mathrm{Reg}_X^{\smile\cap}$,

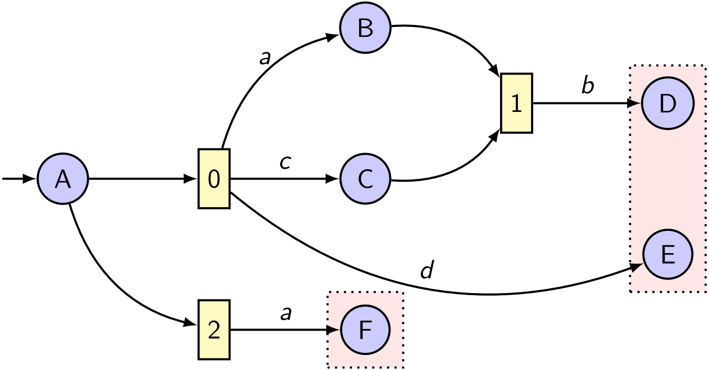
$$\mathrm{Rel} \models e \leqslant f \Leftrightarrow {}^{\blacktriangleleft}G(e) \subseteq {}^{\blacktriangleleft}G(f)$$

Almost proven in:

H. Andréka, S. Mikulás, and I. Németi. The equational theory of Kleene lattices. *TCS*, 412(52):7099–7108, 2011

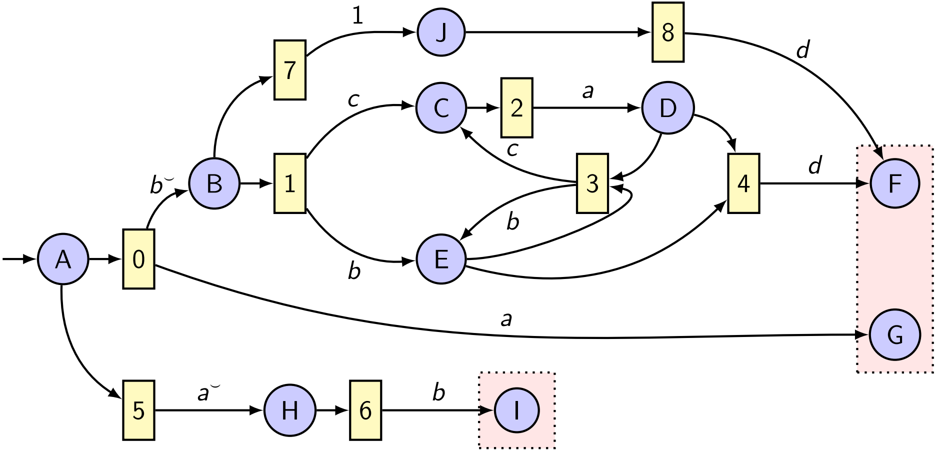
Example

$$(((a \cap c) \cdot b) \cap d) \cup a$$

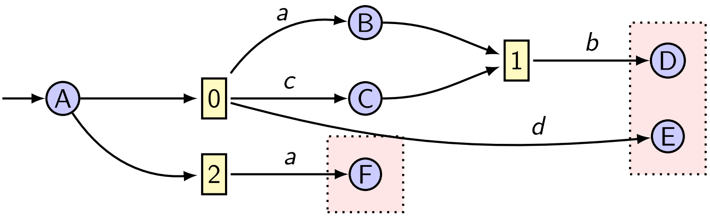


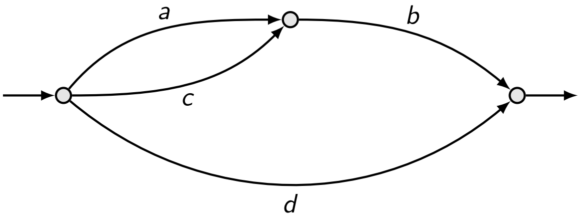
Example

$$\left(b^{\smile} \cdot (a \cdot c \cap b)^{\star} \cdot d\right) \cap a \;\cup\; a^{\smile} \cdot b$$



Reading a graph in an automaton





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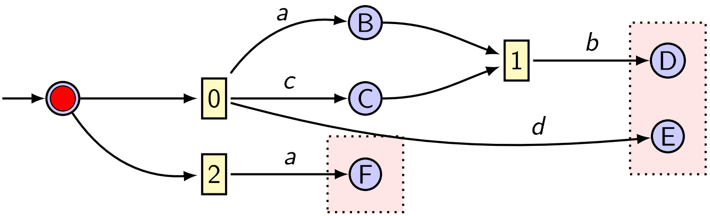

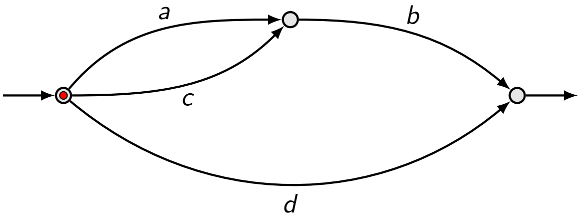

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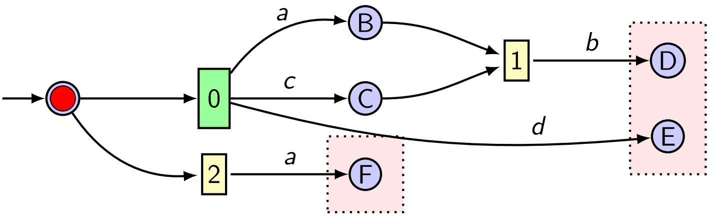

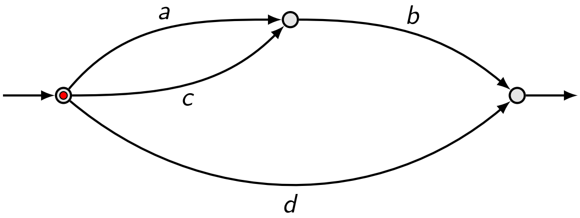

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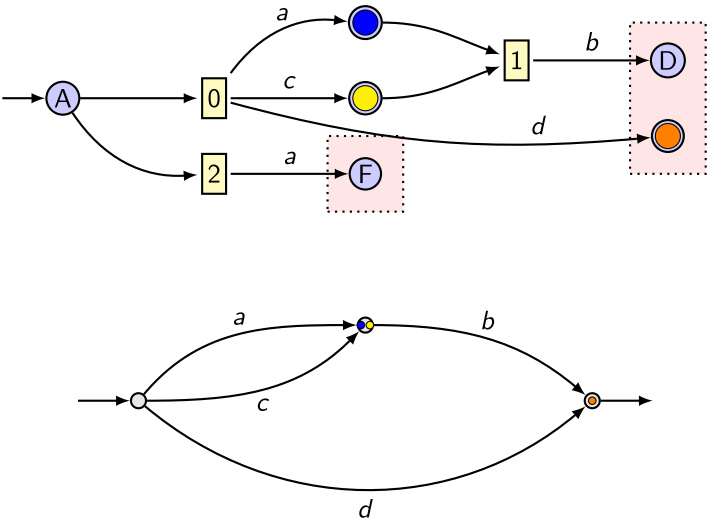

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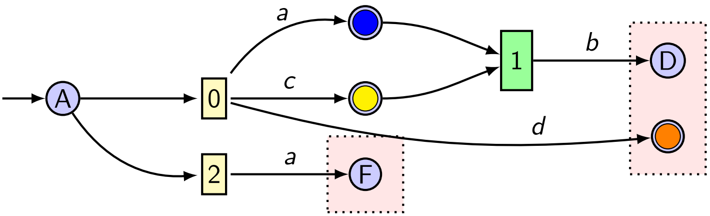

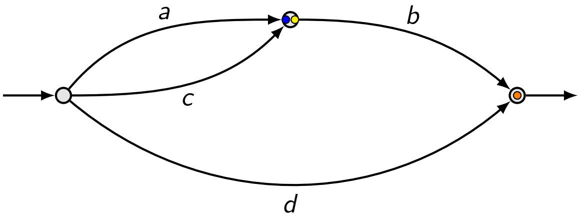

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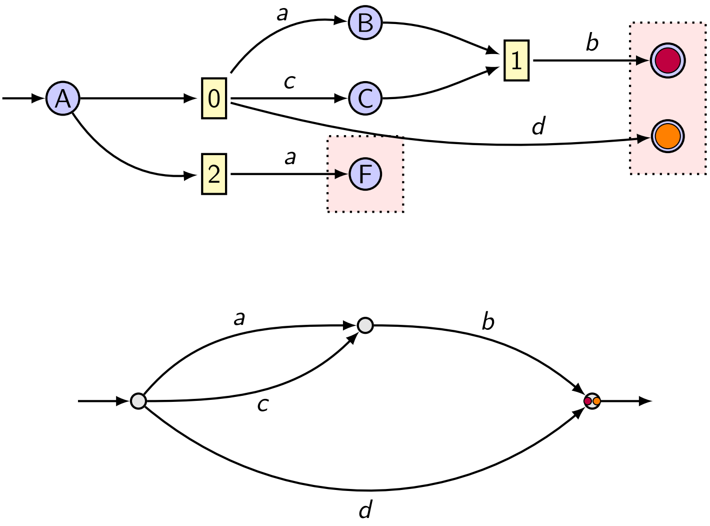
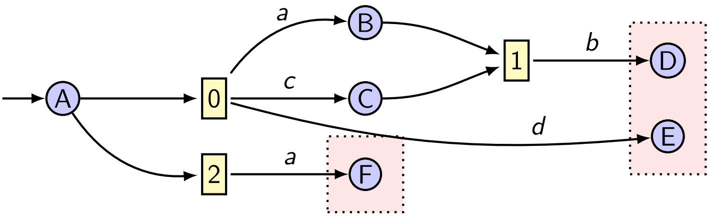
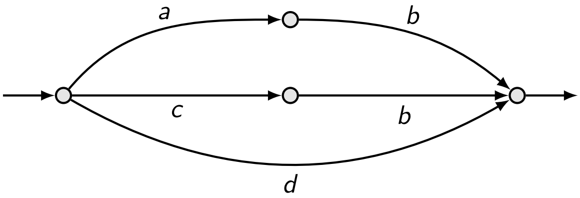


Success!

Reading a graph in an automaton





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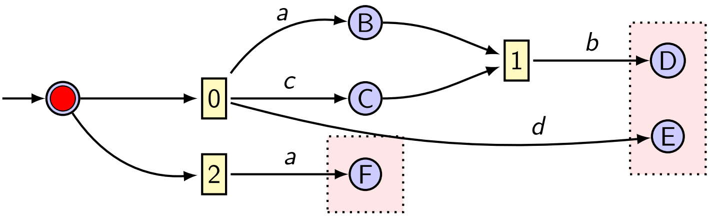

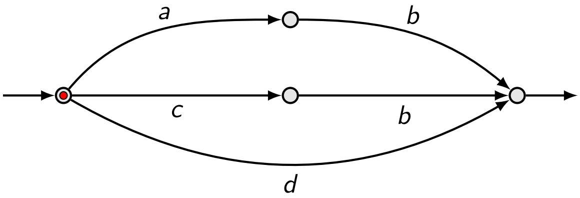

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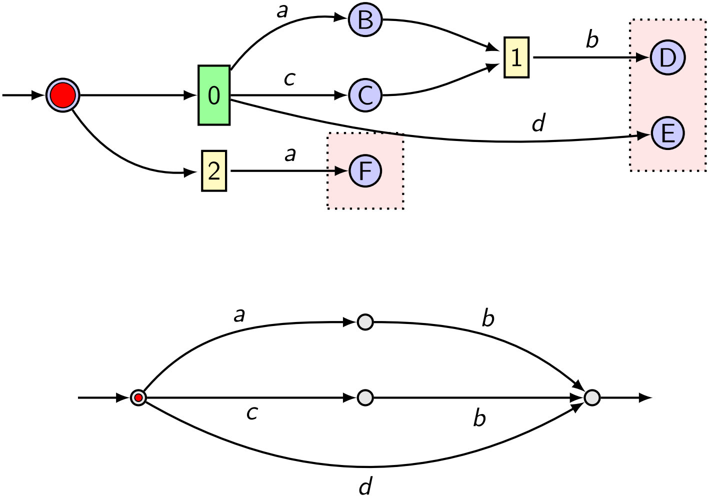

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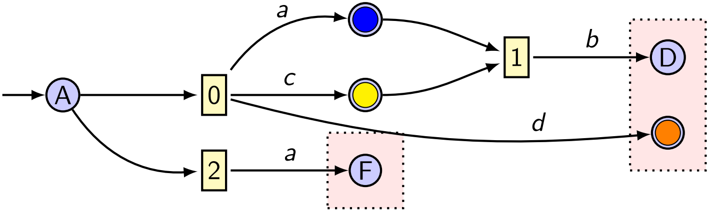

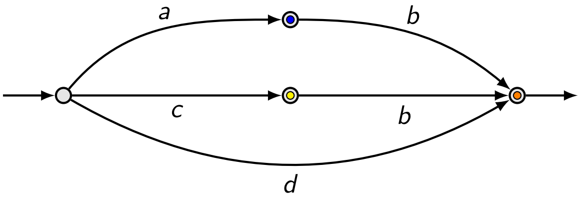

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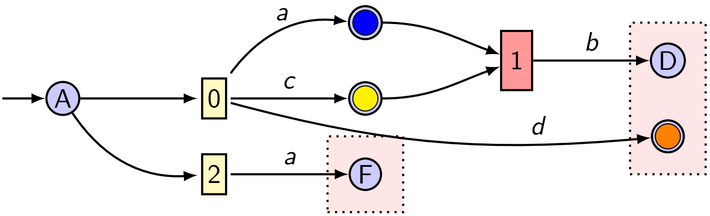
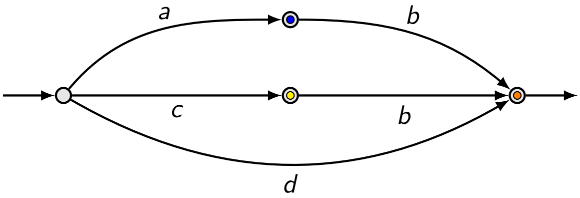
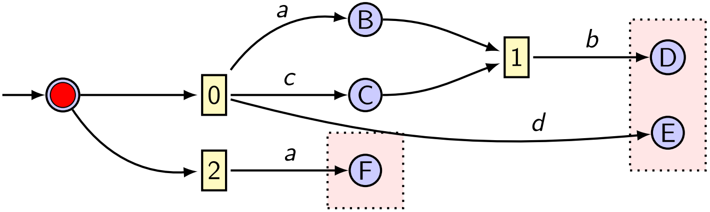
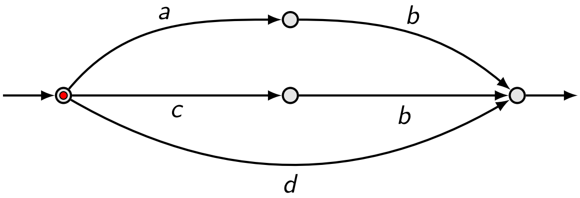




Failure!

Reading a graph in an automaton





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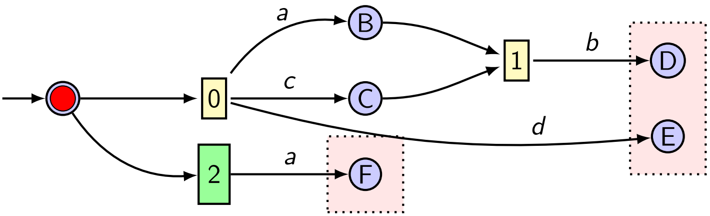

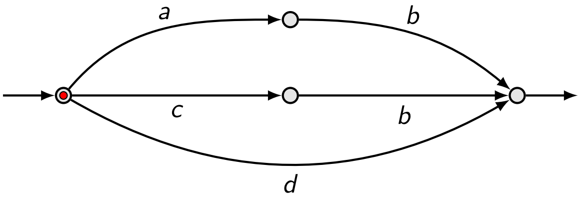

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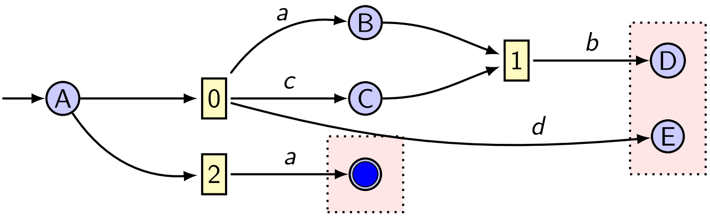

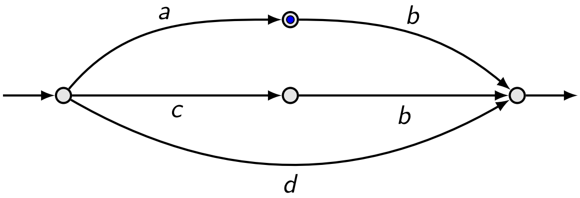

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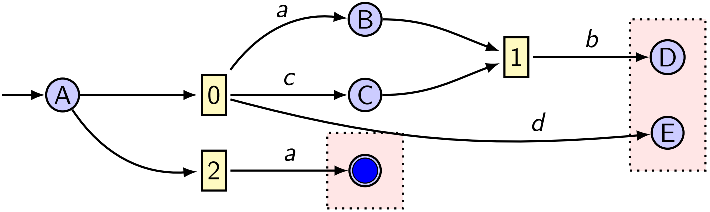

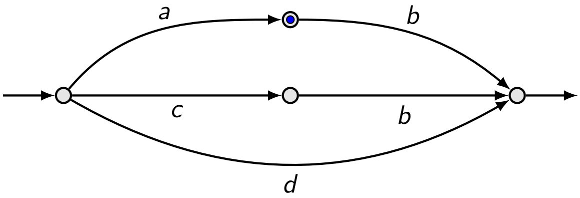

Failure!

Language recognised by an automaton

Correctness

For any $e \in \mathrm{Reg}_X^{\smile\cap}$,

$$e$$

Language recognised by an automaton

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For any $e \in \mathrm{Reg}_X^{\smile\cap}$,

$$\mathscr{A}(e)$$

Language recognised by an automaton

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For any $e \in \mathrm{Reg}_X^{\smile\cap}$,

$$\mathscr{L}\left(\mathscr{A}\left(e\right)\right)$$

Language recognised by an automaton

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For any $e \in \mathrm{Reg}_X^{\smile\cap}$,

$$\mathscr{L}\left(\mathscr{A}(e)\right) = {}^{\blacktriangleleft}G\left(e\right).$$

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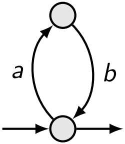
This far:

$e, f \in \mathrm{Reg}_X^{\smile\cap}$

$$\mathrm{Rel} \models e \leqslant f \;\Leftrightarrow\; {}^{\blacktriangleleft}G(e) \subseteq {}^{\blacktriangleleft}G(f) \;\Leftrightarrow\; \mathscr{L}(\mathscr{A}(e)) \subseteq \mathscr{L}(\mathscr{A}(f)).$$

Restriction: identity-free lattice terms

$G\left((a \cdot b) \cap 1\right)$:



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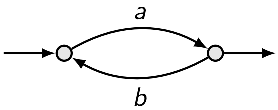

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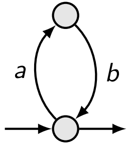

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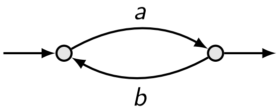

$$u, v \in \mathrm{W}_X^{-} ::= 0 \mid 1 \mid x \in X \mid u \cdot v \mid u \cap v \mid u \cup v \mid u^{\star} \mid u^{\smile}$$

Identity-free Kleene Lattice

$$e, f \in \mathrm{Reg}_X^{\cap -} ::= 0 \mid 1 \mid x \in X \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{+} \mid e^{\smile}$$

Conclusion and future work

Results

- Reduction of relational equivalence to equality of closed graph languages.

Ongoing/Future work

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- Representation of closed graph languages through Petri automata.

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This decision procedure was implemented in OCaml, and is available as an online application.

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- Converting back Petri automata into expressions.

Conclusion and future work

Results

- Reduction of relational equivalence to equality of closed graph languages.
- Representation of closed graph languages through Petri automata.
- Decidability of simple automata equivalence, thus of relational equivalence for Identity-free Kleene Lattices.
- Simple Petri automata equivalence is EXPSPACE-complete.

This decision procedure was implemented in OCAML, and is available as an online application.

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- Decidability with 1 and/or $\_^{\smile}$.

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- Completeness.

That's it!

Thank you !

The content presented here has been accepted for publication in LICS 2015.

`http://perso.ens-lyon.fr/paul.brunet/rklm`.

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