A complete axiomatisation of a fragment of language algebra

CSL

January 2020

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University College London

UCL

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Proved and developped in Coq

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Syntax and semantics

Expressions

$$e, f \in \mathbb{E}_X ::= 1 \mid 0 \mid x \mid e \cdot f \mid e + f \mid e^+ \mid e \cap f \mid \overline{e}$$

Interpretations

Given $\sigma : X \to \mathcal{P}(\Sigma^\star)$ we can define $[\![-]\!] : \mathbb{E}_X \to \mathcal{P}(\Sigma^\star)$:

$$[\![1]\!] \coloneqq \{\varepsilon\} \qquad [\![0]\!] \coloneqq \emptyset$$

$$[\![x]\!] \coloneqq \sigma(x) \qquad [\![e \cdot f]\!] \coloneqq [\![e]\!] \cdot [\![f]\!]$$

$$[\![e + f]\!] \coloneqq [\![e]\!] \cup [\![f]\!] \qquad [\![e^+]\!] \coloneqq [\![e]\!]^+ = \bigcup_{n>0} [\![e]\!]^n$$

$$[\![e \cap f]\!] \coloneqq [\![e]\!] \cap [\![f]\!] \qquad [\![\overline{e}]\!] \coloneqq \overline{[\![e]\!]} = \{\overline{u} \mid u \in L\}$$

(Mirror image a.k.a. reverse: $\overline{abc} = cba$.)

Universal laws

$\mathrm{Lang} \models e = f$

$e, f \in \mathbb{E}_X$ are equivalent iff $\forall \sigma, [\![e]\!] = [\![f]\!]$.

This relation is decidable and ExpSpace-complete.

PB, "Reversible Kleene Lattices", MFCS 2017

Can we find a "good" set of axioms that is sound and complete for language algebra?

$$Ax \vdash e = f \quad \underset{\text{completeness}}{\overset{\text{soundness}}{\rightleftarrows}} \quad \mathrm{Lang} \models e = f$$

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$$Ax \vdash e = f \quad \underset{\text{completeness}}{\overset{\text{soundness}}{\rightleftarrows}} \quad \text{Lang} \models e = f$$

$$e \equiv f \qquad\qquad e \simeq f$$

Related work

Reversible Kleene Lattices

$0, 1, \cdot, +, \cap, \_^+, \overline{\_}$

This paper

Related work

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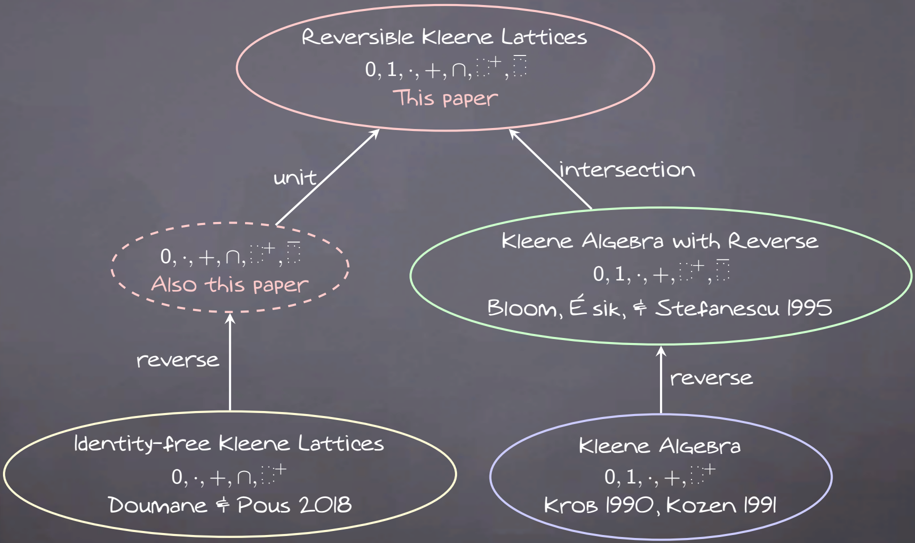
Overview



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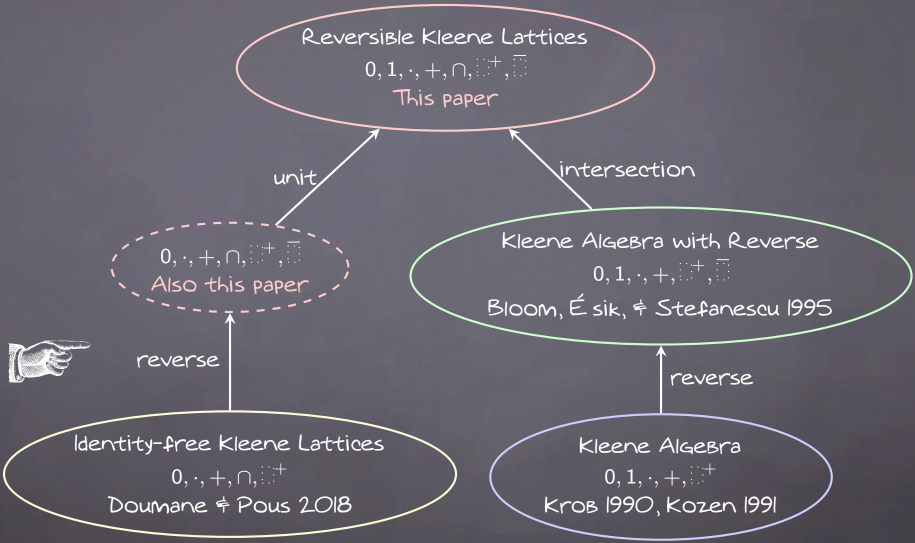
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Identity-free Kleene Lattices

Completeness for Identity-free Kleene Lattices

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Abstract

We provide a finite set of axioms for identity-free Kleene lattices, which we prove sound and complete for the equational theory of their relational models. Our proof builds on the completeness theorem for Kleene algebra, and on a novel automata construction that makes it possible to extract axiomatic proofs using a Kleene-like algorithm.

Identity-free Kleene Lattices

Distributive Lattice

$$e + f = f + e \quad (1)$$
$$e + (f + g) = (e + f) + g \quad (2)$$
$$e + 0 = e \quad (3)$$
$$e \cap f = f \cap e \quad (4)$$
$$e \cap e = e \quad (5)$$
$$e \cap (f \cap g) = (e \cap f) \cap g \quad (6)$$
$$(e + f) \cap g = e \cap g + f \cap g \quad (7)$$
$$(e \cap f) + e = e \quad (8)$$

Concatenation & iteration

$$e \cdot (f \cdot g) = (e \cdot f) \cdot g \quad (9)$$
$$e \cdot 0 = 0 = 0 \cdot e \quad (10)$$
$$(e + f) \cdot g = e \cdot g + f \cdot g \quad (11)$$
$$e \cdot (f + g) = e \cdot f + e \cdot g \quad (12)$$
$$e^+ = e + e \cdot e^+ \quad (13)$$
$$e^+ = e + e^+ \cdot e \quad (14)$$
$$e \cdot f + f = f \Rightarrow e^+ \cdot f + f = f \quad (15)$$
$$f \cdot e + f = f \Rightarrow f \cdot e^+ + f = f \quad (16)$$

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$e \equiv f \overset{\text{Doumane \& Pous 2018}}{\Longleftrightarrow} \mathsf{Rel} \models e = f$

Identity-free Kleene Lattices

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$e \equiv f$ $\xleftrightarrow{\text{Doumane \& Pous 2018}}$ $\mathrm{Rel} \models e = f$ $\xleftrightarrow{\text{Andréka, Mikulás, \& Németi 2011}}$ $\mathrm{Lang} \models e = f$

THE EQUATIONAL THEORY OF KLEENE LATTICES

HAJNAL ANDRÉKA, SZABOLCS MIKULÁS, AND ISTVÁN NÉMETI

Abstract. Languages and families of binary relations are standard interpretations of Kleene algebras. It is known that the equational theories of these interpretations coincide and that the free Kleene algebra is representable both as a relational and as a language algebra. We investigate the identities valid in these interpretations when we expand the signature of Kleene algebras with the meet operation. In both cases meet is interpreted as intersection. We prove that in this case there are more identities valid in language algebras than in relational algebras (exactly three more in some sense), and representability of the free algebra holds for the relational interpretation but fails for the language interpretation. However, if we exclude the identity constant from the algebras when we add meet, then the equational theories of the relational and language interpretations remain the same, and the free algebra is representable as a language algebra, too. The moral is that only the identity constant behaves differently in the language and the relational interpretations, and only meet makes this visible.

Reverse

Warning!

$$\mathsf{Rel} \models e = f \not\Leftrightarrow \mathsf{Lang} \models e = f.$$

In particular:

- ☞ $\mathsf{Rel} \models a \leq a \cdot \overline{a} \cdot a$
- ☞ $\mathsf{Lang} \not\models a \leq a \cdot \overline{a} \cdot a.$

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In particular:

- ☞ $\text{Rel} \models a \leq a \cdot \overline{a} \cdot a$
- ☞ $\text{Lang} \not\models a \leq a \cdot \overline{a} \cdot a.$

$$\overline{\overline{e}} = e \tag{17}$$

$$\overline{e + f} = \overline{e} + \overline{f} \tag{18}$$

$$\overline{e \cdot f} = \overline{f} \cdot \overline{e} \tag{19}$$

$$\overline{e \cap f} = \overline{e} \cap \overline{f} \tag{20}$$

$$\overline{e^{+}} = \overline{e}^{+} \tag{21}$$

Reverse

Rewrite system

$$e \mapsto e'$$

$$\overline{0} \mapsto 0 \qquad \overline{\overline{e}} \mapsto e$$

$$\overline{e \cup f} \mapsto \overline{e} \cup \overline{f} \qquad \overline{e \cap f} \mapsto \overline{e} \cap \overline{f}$$

$$\overline{e \cdot f} \mapsto \overline{f} \cdot \overline{e} \qquad \overline{e^+} \mapsto \overline{e}^+$$

e' is a term without $\overline{\square}$ over alphabet $X' := X \cup \{\overline{x} \mid x \in X\}$.

Theorem

$e \equiv f$ Lang $\models e = f$

$e' \equiv f'$ Lang $\models e' = f'$

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Theorem

$$\begin{array}{ccc} e \equiv f & & \text{Lang} \models e = f \\ \Updownarrow & & \Updownarrow \\ e' \equiv f' & \Leftrightarrow & \text{Lang} \models e' = f' \end{array}$$

Reverse

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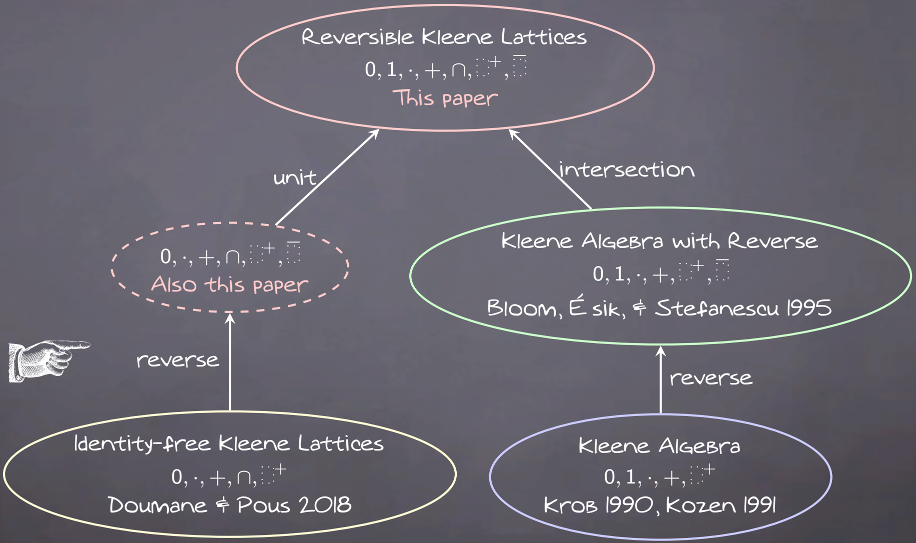
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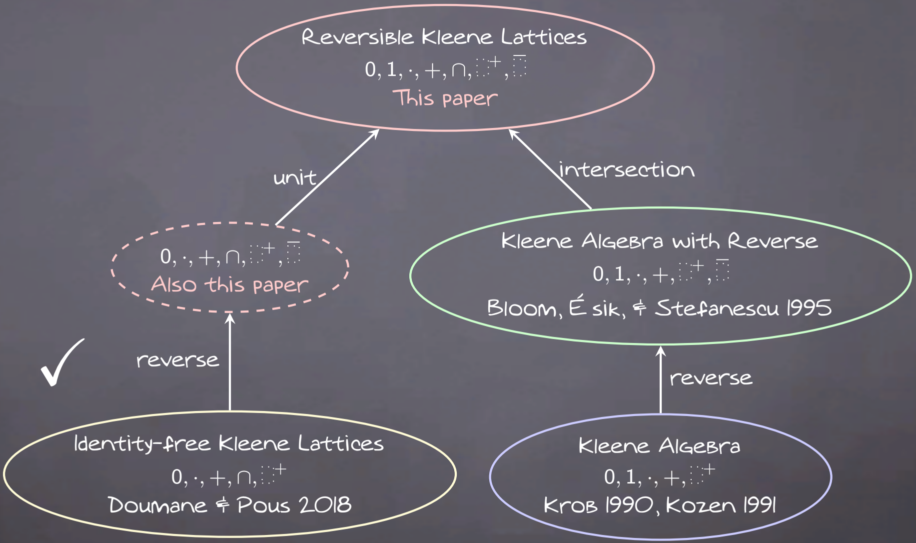
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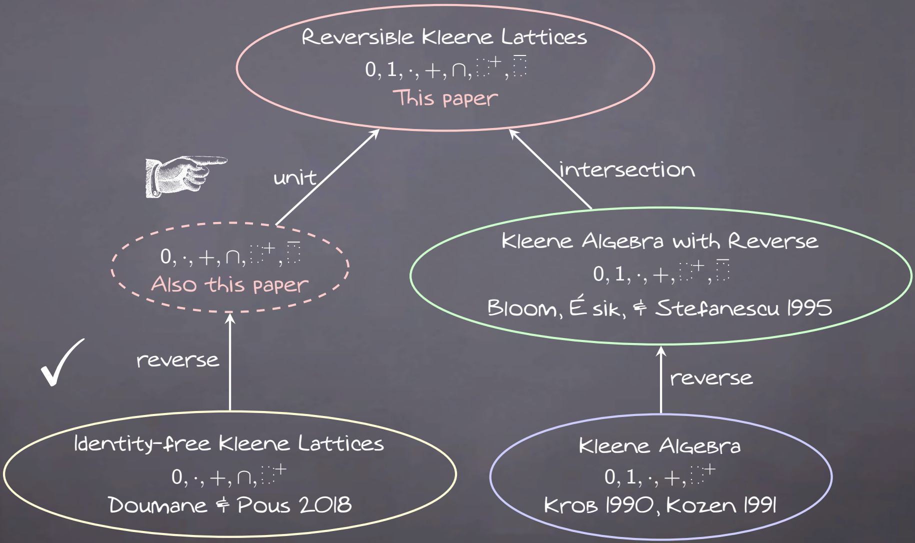

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New axioms

$$1 \cdot e = e = e \cdot 1 \tag{22}$$

$$1 \cap (e \cdot f) = 1 \cap (e \cap f) \tag{23}$$

$$1 \cap \overline{e} = 1 \cap e \tag{24}$$

$$(1 \cap e) \cdot f = f \cdot (1 \cap e) \tag{25}$$

$$((1 \cap e) \cdot f) \cap g = (1 \cap e) \cdot (f \cap g) \tag{26}$$

$$(g + (1 \cap e) \cdot f)^+ = g^+ + (1 \cap e) \cdot (g + f)^+ \tag{27}$$

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Soundness

$$e \equiv f \Rightarrow e \simeq f.$$

Step 1 - Normal forms

$$e \equiv \sum_i (1 \cap A_i) \cdot e_i$$

where

- ☞ A_i are intersections of variables;
- ☞ e_i are either exactly 1 or 1-free;

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Step I - Normal forms

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where

- A_i are intersections of variables;
- e_i are either exactly 1 or 1-free;

$$e \lesssim f \stackrel{?}{\Rightarrow} e \leqq f$$

reduces to

$$(1 \cap A_i) \cdot e_i \lesssim f \stackrel{?}{\Rightarrow} (1 \cap A_i) \cdot e_i \leqq f$$

Step II - Removing tests on the left

Given $A \subseteq X$ and a term f, we define $\langle f \rangle_A$ by substituting every $a \in A$ with $1 + a$ in f.

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Lemma

$$(1 \cap A) \cdot \langle f \rangle_A \leqq f \leqq \langle f \rangle_A.$$

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Lemma

$$(1 \cap A) \cdot \langle f \rangle_A \leqq f \leqq \langle f \rangle_A.$$

Lemma

$$(1 \cap A) \cdot e \lesssim f \Rightarrow e \lesssim \langle f \rangle_A$$

Step III - Removing tests on the right

By magic, we can get from a term f a 1-free term $[f]$ such that:

Lemma

1) $[f] \leqq f$;
2) for any 1-free e, if $e \lesssim f$, then $e \lesssim [f]$.

Main result

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Theorem

$$e \equiv f \Leftrightarrow e \simeq f.$$

That's all folks!

Thank you!

See more at:
http://paul.brunet-zamansky.fr