The Theory of Languages

Highlights in London

September 12-15, 2017

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University College London

Universal laws

$$a \cup b = b \cup a \qquad \text{(commutativity of union)}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{(associativity of concatenation)}$$

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Variables in X

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Mirror image

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Intersection

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Language equivalence

$\mathrm{Lang} \models e \simeq f$ iff $\forall\Sigma, \forall\sigma : X \to \mathcal{P}\left(\Sigma^\star\right), \widehat{\sigma}\left(e\right) = \widehat{\sigma}\left(f\right)$.

representation

$$r : \mathbb{E}_X \to R$$

Effective representation

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Computable

Decidable

Effective free representation

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$$\textcolor{red}{Ax} \vdash e = f \iff r(e) = r(f) \iff \mathrm{Lang} \models e \simeq f$$

Effective free representation

$$r : \mathbb{E}_X \to R$$

Computable → r ; Decidable → R

$$Ax \vdash e \leq f \iff r(e) \sqsubseteq r(f) \iff \mathrm{Lang} \models e \subseteq f$$

Free representations: Kleene Algebra

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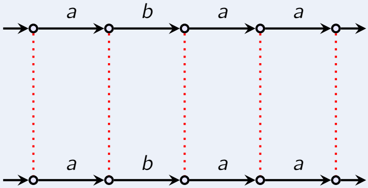
Multiplicative fragment: $s, t ::= 1 \mid e \cdot f$

Words



Equality

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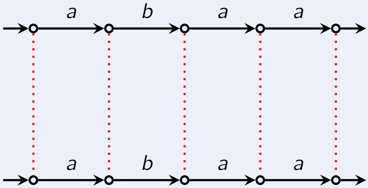
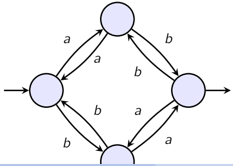


Equality

Add unions: take sets of things

Sets of words = languages

Finite state automata



Free representations: Identity free Kleene Lattices

$e, f ::= 0 \quad | \; e \cdot f \; | \; e \cup f \quad | \; e^+ \; | \; e \cap f$

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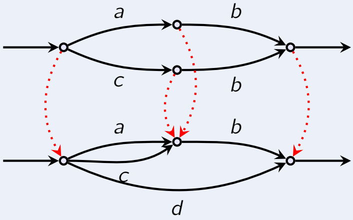
Series parallel graphs

Graph homomorphism



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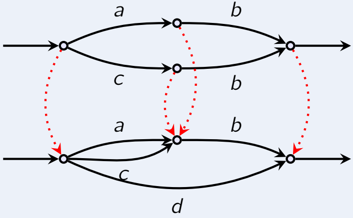
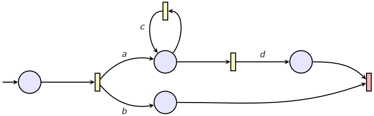


Add unions: take sets of things

Set of graphs = ideals of graph languages

Petri automata



Free representations: Reversible Kleene Lattices

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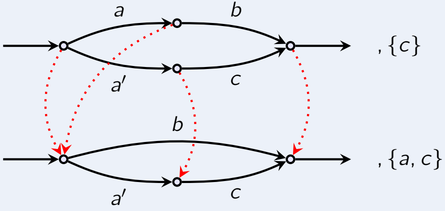
Series parallel graphs with tests

Weak-morphisms



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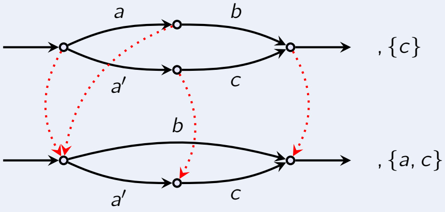
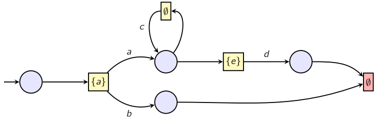


Add unions: take sets of things

Set of weak graphs = ideals of weak graph languages

Weighted Petri automata



Main results

$$e, f \in \mathbb{E}_X ::= 0 \mid 1 \mid a \mid e \cup f \mid e \cap f \mid e \cdot f \mid e^{\smile} \mid e^{\star}.$$

Decidability theorem

The equational theory of languages over $\mathbb{E}_X$ is ExpSpace-complete.

Hardness comes from the universality problem for regular languages with intersection.

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$$e, f \in \mathbb{E}_X^{fin} ::= 0 \mid 1 \mid a \mid e \cup f \mid e \cap f \mid e \cdot f \mid e^{\smile}.$$

Axiomatisability theorem

The equational theory of languages over $\mathbb{E}_X^{fin}$ is finitely axiomatisable.

Open problems

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$$b \leq a \setminus c \qquad \Leftrightarrow \qquad a \cdot b \leq c \qquad \Leftrightarrow \qquad a \leq c \,/\, b$$

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That's all folks!

See more at:
`http://paul.brunet-zamansky.fr`

Outline

Bibliography

Freyd & Scedrov, **Categories, Allegories**, *1990*

Andréka & Bredikhin, **The equational theory of union-free algebras of relations**, *1995*

Andréka, Mikulás & Németi, **The equational theory of Kleene lattices**, *2011*

Bloom, Ésik & Stefanescu, **Notes on equational theories of relations**, *1995*

B. & Pous, **Petri Automata for Kleene Allegories**, *2015*

B., **Reversible Kleene lattices**, *2017*

Weak graphs

Definition

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Weak graph preorder

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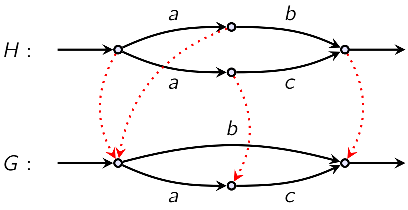
$A = \{a\}$



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Corollary

$$\mathrm{Lang} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v).$$

Free representation of expressions

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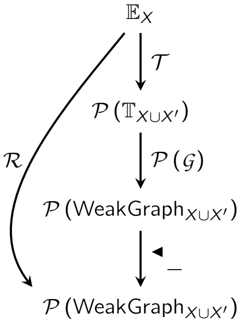

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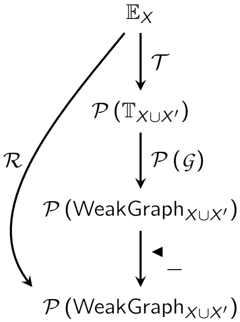

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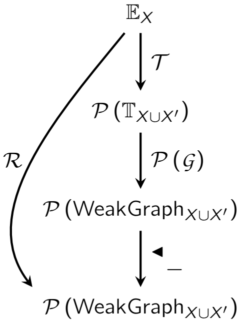

Lemma

If e doesn't use the Kleene star, then $\mathcal{T}(e)$ is finite.

Theorem

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Decidability and Complexity

Representation Theorem

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$\mathcal{L}(\mathcal{A}(e)) = \mathcal{R}(e)$ and $|\mathcal{A}(e)| = |e| \times 2^{|X|}$.

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Main Theorem

The equational theory of languages over the signature $\mathbb{E}_X$ is ExpSpace-complete.

Axiomatization for series parallel terms

$u, v, w \in \mathbb{SP}_X ::= a \mid u \parallel v \mid u \cdot v, \quad A \subseteq X.$

$$\frac{}{A \vdash_{sp} u \leq u} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} v \leq w}{A \vdash_{sp} u \leq w}$$

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Associativity

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Meet-semilattice

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