Deciding Kleene Algebra with converse is PSPACE-complete

Talk at the GeoCal meeting

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ENS de Lyon, Université de Lyon

March 25^{th}, 2014

Introduction

$$(x^\star + y) \cdot z \qquad\qquad (x^\star \cdot z) + (y \cdot z)$$

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$$\forall S, \forall \sigma : \mathcal{R}eg_X \to \mathcal{R}el(S)$$

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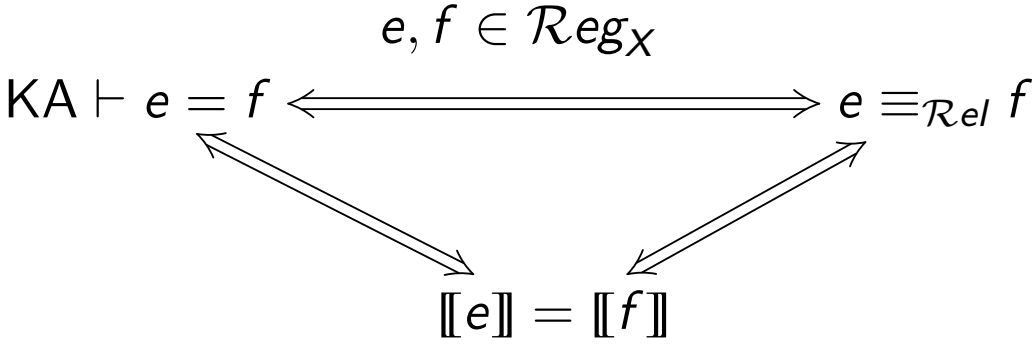

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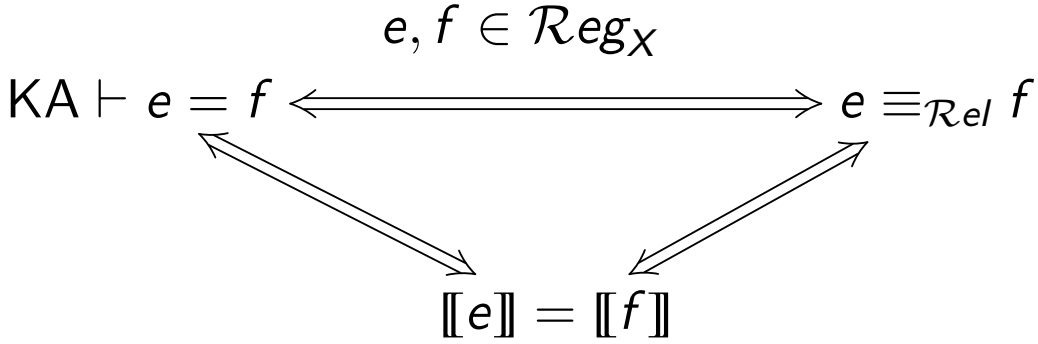
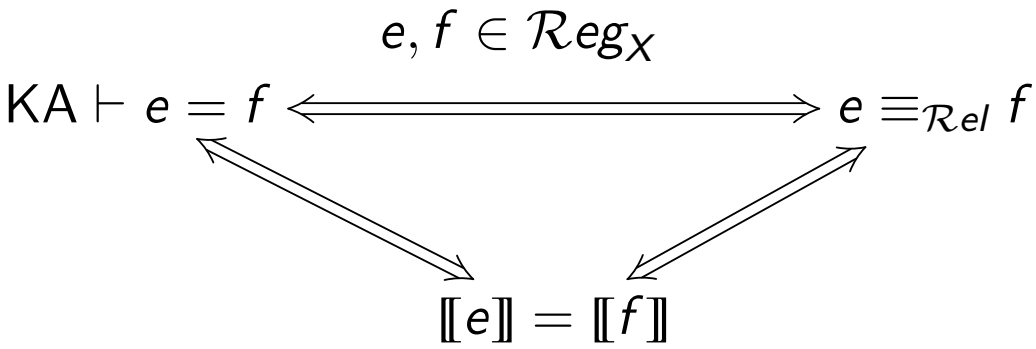
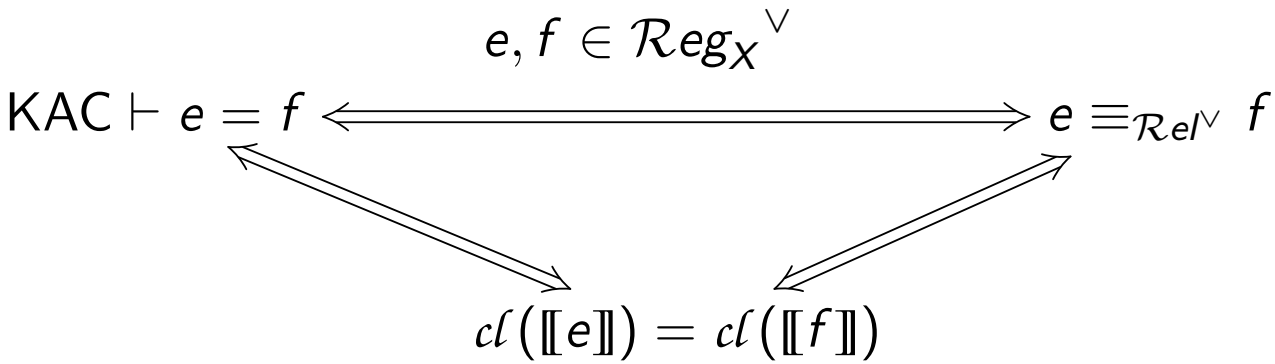


What if we add a *converse* operation to regular expressions ?

Introduction





Introduction

$$e, f \in \mathcal{R}eg_X{}^{\vee}$$

$$e \equiv_{\mathcal{R}el^{\vee}} f$$

$$\Leftrightarrow$$

$$cl([\![e]\!]) = cl([\![f]\!])$$

Plan

Plan

Regular expressions with converse

Regular expressions with converse over X

$$e, f \in \mathcal{R}eg_X{}^{\vee} ::= \mathbb{0} \mid \mathbb{1} \mid x \in X \mid e + f \mid e \cdot f \mid e^{\star} \mid e^{\vee}$$

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Given any map :

$$\sigma : X \longrightarrow \mathcal{R}el(S)\,,$$

we can build uniquely a morphism

$$\hat{\sigma} : \mathcal{R}eg_X{}^{\vee} \longrightarrow \mathcal{R}el(S)\,.$$

Relational equivalence

For $e, f \in \mathcal{R}eg_X{}^{\vee}$:

$$e \equiv_{\mathcal{R}el^{\vee}} f$$

means that

$$\forall S, \forall \sigma : X \to \mathcal{R}el(S), \hat{\sigma}(e) = \hat{\sigma}(f).$$

The equational theory KAC

The equational theory KAC of regular algebras with converse over binary relations consists of the axioms of KA together with the following :

$$(a+b)^\vee = a^\vee + b^\vee \tag{1}$$

$$(a \cdot b)^\vee = b^\vee \cdot a^\vee \tag{2}$$

$$(a^*)^\vee = (a^\vee)^* \tag{3}$$

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$$aa^\vee a \geqslant a \tag{5}$$

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From $\mathcal{R}eg_X{}^\vee$ to $\mathcal{R}eg_{\mathbf{X}}$

Let X be a finite alphabet. For $e \in \mathcal{R}eg_X$, we write $[\![e]\!] \subseteq X^*$ for the *language denoted by* e.

- $X' := \{x' \mid x \in X\}$ is a disjoint copy of X,
- and $\mathbf{X} := X \cup X'$.

We apply the following rewriting system :

$$\begin{cases} (a+b)^\vee \longmapsto a^\vee + b^\vee \\ (a \cdot b)^\vee \longmapsto b^\vee \cdot a^\vee \\ (a^*)^\vee \longmapsto (a^\vee)^* \\ a^{\vee\vee} \longmapsto a \end{cases}$$

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We get $\mathbf{e} \in \mathcal{R}eg_{\mathbf{X}}$.

$$e, f \in \mathcal{R}eg_X \quad : \quad e \equiv_{\mathcal{R}el} f \quad \begin{matrix} \Rightarrow \\ \Leftarrow \end{matrix} \quad [\![e]\!] = [\![f]\!]$$

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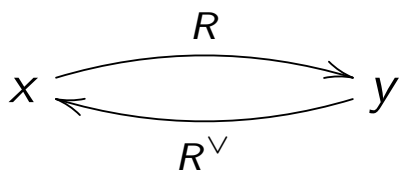

$e = aa^\vee a, f = a$:

$$[\![\mathbf{e}]\!] = \{aa'a\} \not\supseteq \{a\} = [\![\mathbf{f}]\!]$$

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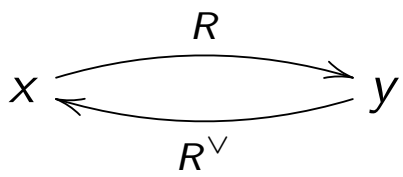

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Reduction relation and closure

Converse for words : $\overline{w}$

For a word $w \in \mathbf{X}^*$, we define inductively $\overline{w}$:

$$\begin{array}{ll} \forall x \in X, & \overline{x} := x' \\ \forall x' \in X', & \overline{x'} := x \end{array} \quad \left| \quad \begin{array}{c} \overline{\epsilon} := \epsilon \\ \overline{wx} := \overline{x}\ \overline{w}. \end{array} \right.$$

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Redution relation : $u \rightsquigarrow v$

$$u_1 \cdot w\overline{w}w \cdot u_2 \quad \rightsquigarrow \quad u_1 \cdot w \cdot u_2$$

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Closure of a language : $cl(L)$

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Example :

$aa'a = a\bar{a}a \rightsquigarrow a$, so we have : $\mathcal{cl}(\{aa'a\}) = \{a, aa'a\} \supseteq \{a\}$.

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Closure

Theorem [a]

a. Bloom, S. L., Ésik, Z., and Stefanescu, G. (1995). Notes on equational theories of relations.
Algebra Universalis, 33(1) :98–126

$$e \equiv_{\mathcal{R}el^{\vee}} f \quad \Leftrightarrow \quad cl([\![\mathbf{e}]\!]) = cl([\![\mathbf{f}]\!])$$

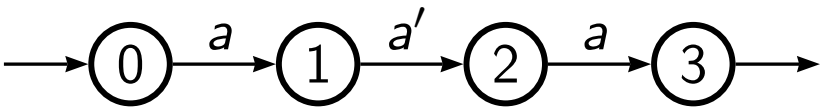
Plan

Problem

Input : an automaton $\mathscr{A}$ over $\mathbf{X}$

Output : an automaton $\mathscr{A}'$ over $\mathbf{X}$ such that $L(\mathscr{A}') = cl(L(\mathscr{A}))$.

Intuition



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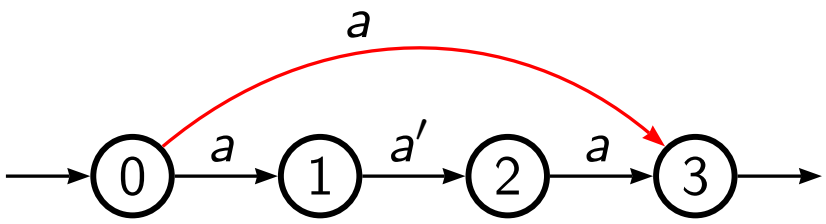

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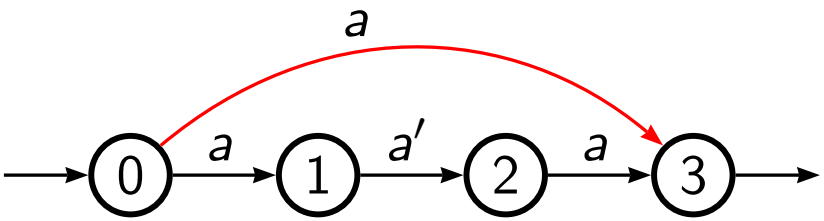

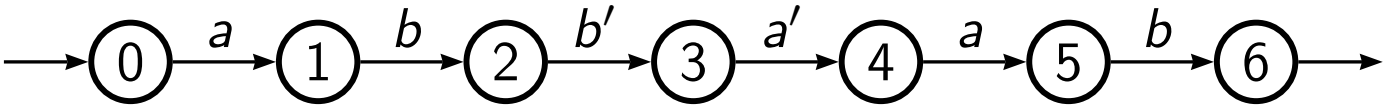

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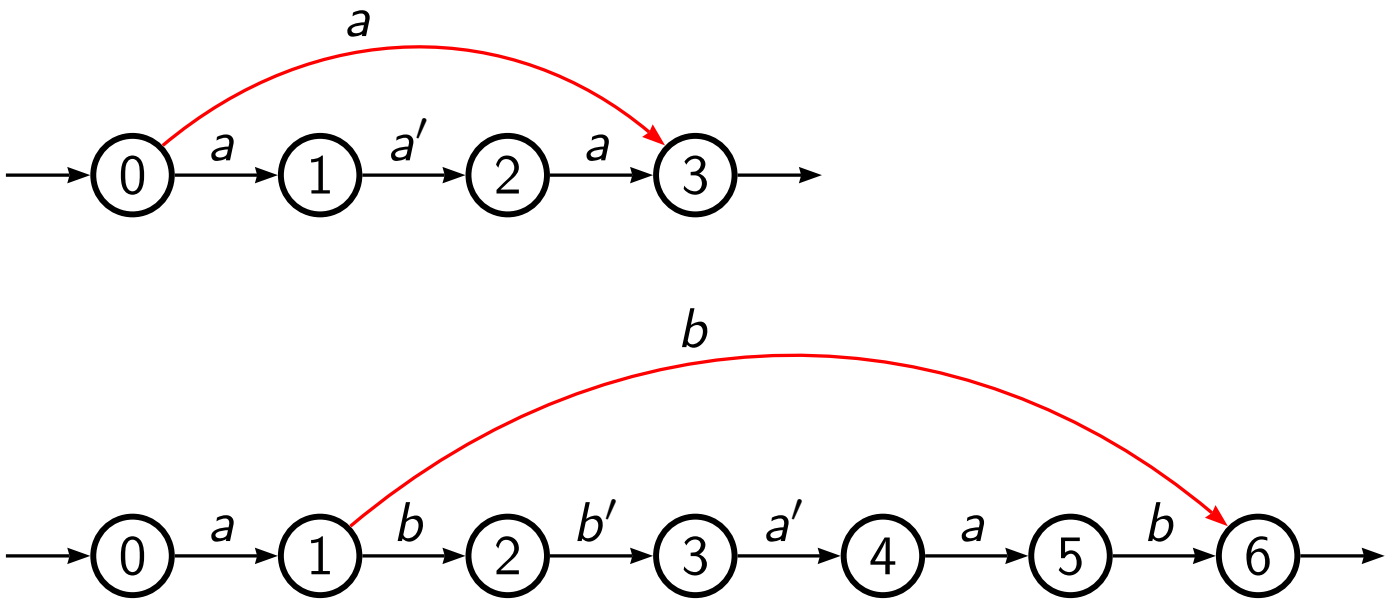

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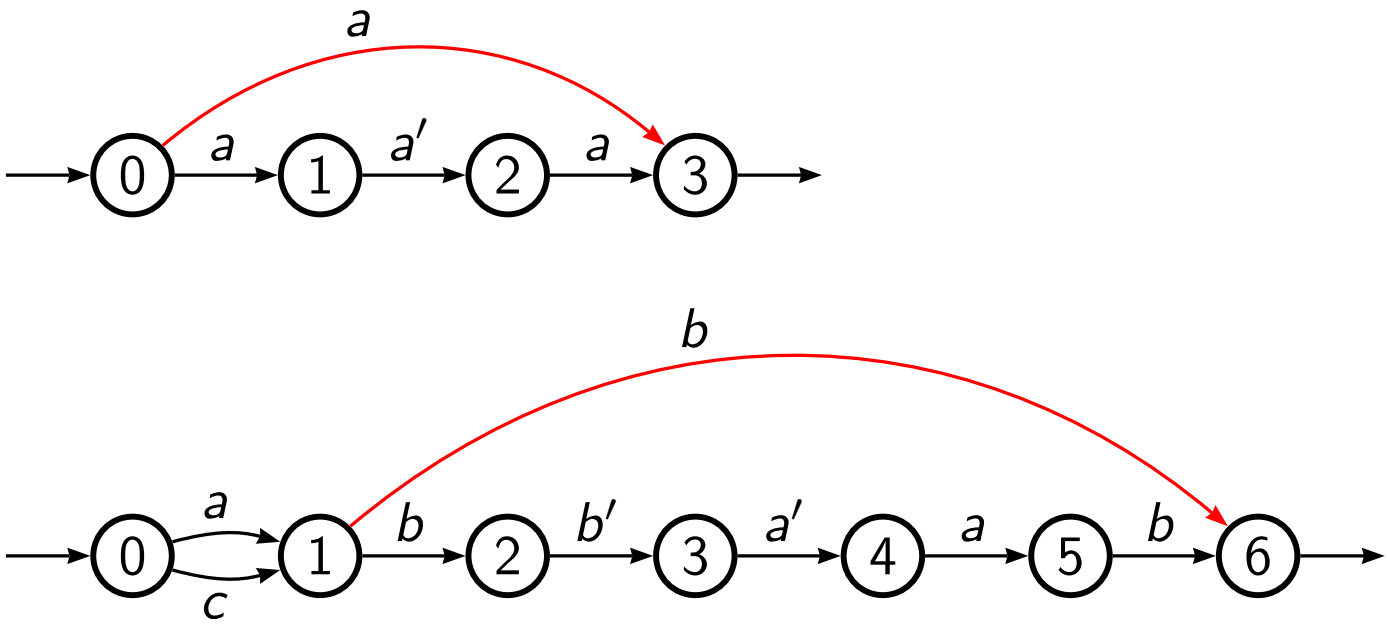

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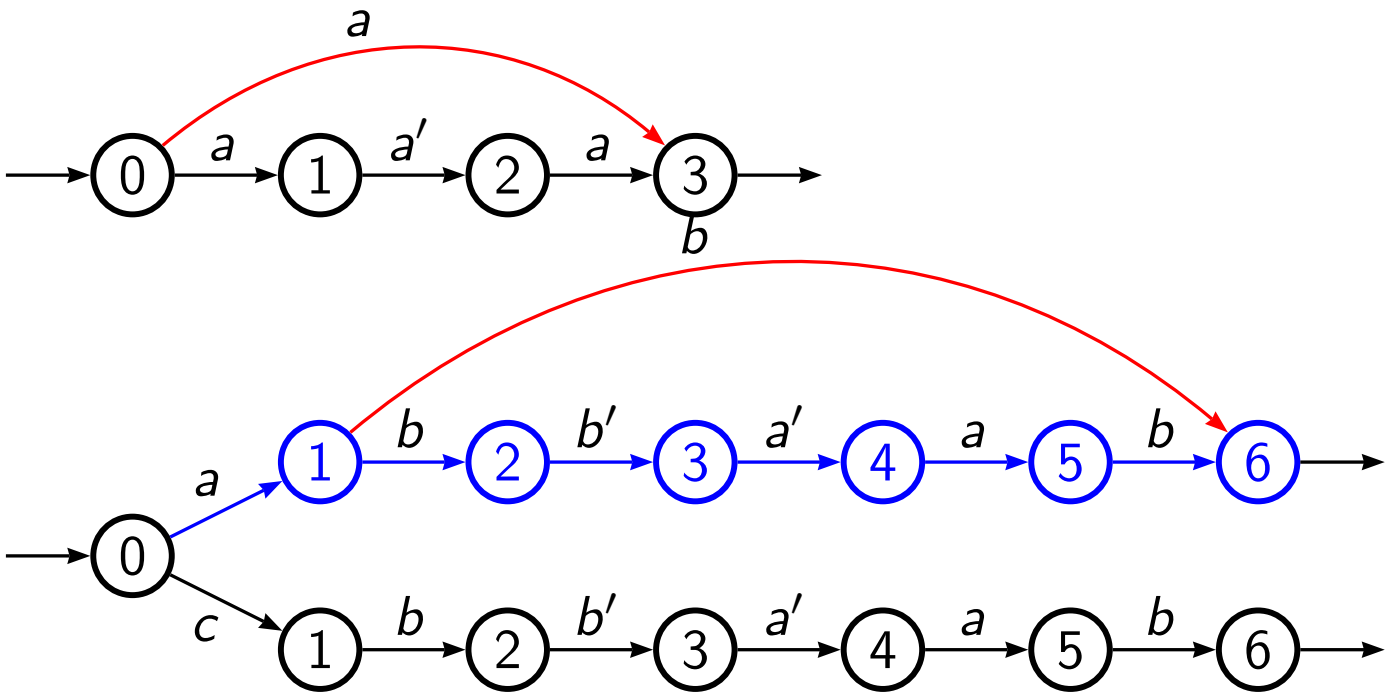

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  - $H \xmapsto{a} H'$,
  - and H' allows to jump from q_3 to q_2.

$\Gamma(w)$

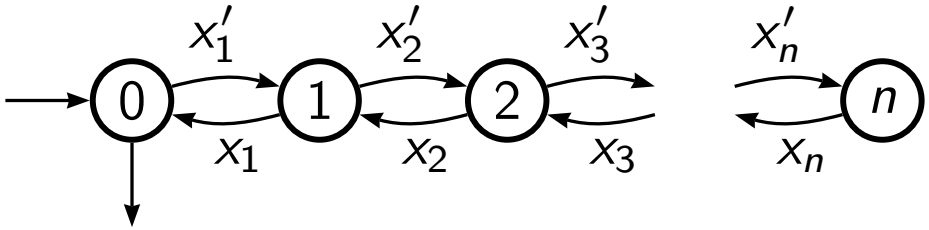
Definition : $\Gamma(w)$

$$\Gamma(\epsilon) = \{\epsilon\}$$
$$\Gamma(wx) = (\{x'\} \cdot \Gamma(w) \cdot \{x\})^*$$

Lemma

$$u \in \Gamma(w) \Leftrightarrow \exists v \in \text{suffixes}(w) : u \rightsquigarrow^* \overline{v}v$$

$\Gamma(x_n \cdots x_1)$ is recognised by the automaton :



$\gamma(w)$

Consider an automaton $\mathscr{A} = \langle Q, A, I, T, \Delta \rangle$, we write $\Delta_x := \{(p, q) \mid p \xrightarrow{x} q \in \Delta\}$.

Definition : $\gamma(w)$

$$\gamma(\epsilon) = \mathrm{Id}_Q$$
$$\gamma(wx) = (\Delta_{x'} \circ \gamma(w) \circ \Delta_x)^*$$

Lemma

$$(p, q) \in \gamma(w) \Leftrightarrow \exists u \in \Gamma(w) : p \xrightarrow{u} q$$
$$\Leftrightarrow \exists u : \exists v \in \mathrm{suffixes}(w) : p \xrightarrow{u} q \wedge u \rightsquigarrow^* \overline{v} v$$

Histories

The set of histories is $G := \{r \in \mathcal{R}el(Q) \mid \exists w \in \mathbf{X}^* : r = \gamma(w)\}$.

Closure Automaton

$cl(\mathscr{A})$

$cl(\mathscr{A}) := \langle Q \times G, \mathbf{X}, I \times \gamma(\epsilon), F \times G, \Delta' \rangle$ with transitions Δ' :

$$(q_1, \gamma(w)) \xrightarrow{x}_{cl(\mathscr{A})} (q_2, \gamma(wx)) \text{ if } (q_1, q_2) \in \Delta_x \circ \gamma(wx)$$

Theorem

$$L(cl(\mathscr{A})) = cl(L(\mathscr{A}))$$

Size

$$\Delta' : \{((q_1, \gamma(w)), x, (q_2, \gamma(wx))) \mid (q_1, q_2) \in \Delta_x \circ \gamma(wx)\}$$

We can see that this construction produces a non-deterministic automaton of size at most $n \times 2^{n \times (n-1)}$.

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Furthermore, it can be easily determinized :

$$\delta' : ((Q_1, \gamma(w)), x) \mapsto (Q_1 \cdot (\Delta_x \circ \gamma(wx)), \gamma(wx))$$

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This deterministic automaton has at most $2^n \times 2^{n \times (n-1)} = 2^{n^2}$ states, which is significantly smaller than $2^{2^{n^2}}$, the size of the automaton from the original construction.

Plan

Automaton equivalence

Let $\mathscr{A}$ and $\mathscr{B}$ be two deterministic automata over some alphabet Σ.

Theorem

$L(\mathscr{A}) \neq L(\mathscr{B}) \Leftrightarrow \exists w \in (L(\mathscr{A}) \ominus L(\mathscr{B})) : |w| \leqslant |\mathscr{A}| \times |\mathscr{B}|.$

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input : $\mathscr{A}_1 = \langle Q_1, \Sigma, i_1, T_1, \delta_1 \rangle$
input : $\mathscr{A}_2 = \langle Q_2, \Sigma, i_2, T_2, \delta_2 \rangle$
output: A Boolean, saying whether or not $\mathscr{A}_1$ and $\mathscr{A}_2$ recognise the same language.

```
1  N ← (|Q1| × |Q2|);
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Automaton equivalence

Let $\mathscr{A}$ and $\mathscr{B}$ be two deterministic automata over some alphabet Σ.

Theorem

$$L(\mathscr{A}) \neq L(\mathscr{B}) \Leftrightarrow \exists w \in (L(\mathscr{A}) \ominus L(\mathscr{B})) : |w| \leqslant |\mathscr{A}| \times |\mathscr{B}|.$$

input : $\mathscr{A}_1 = \langle Q_1, \Sigma, i_1, T_1, \delta_1 \rangle$
input : $\mathscr{A}_2 = \langle Q_2, \Sigma, i_2, T_2, \delta_2 \rangle$
output: A Boolean, saying whether or not $\mathscr{A}_1$ and $\mathscr{A}_2$ recognise the same language.

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A PSPACE algorithm for KAC

input : Two regular expressions with converse $e, f \in \mathcal{R}eg_X{}^\vee$
output: A Boolean, saying whether or not KAC $\vdash e = f$.

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3 $N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$;
4 $((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$;
5 **while** $N > 0$ **do**
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A PSPACE algorithm for KAC

Let's write n and m for the sizes of e and f.

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$\mathcal{O}(n+m)$

$\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2 + m^2\right)$

$\sim log(n) + n^2 + log(m) + m^2$
$\sim \mathcal{O}\left(n^2 + m^2\right)$

$\mathcal{O}(log(n))$

$\mathcal{O}\left(n^2 + m^2\right)$

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So we get a space complexity $\mathcal{O}\left(n^2 + m^2\right)$.

So far

- New construction for deciding KAC.

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- Toy implementation in OCAML of both constructions.
- COQ proof of confluence of the relation $\rightsquigarrow$.

Further work

- Simpler proof of $\mathcal{cl}([\![\mathbf{e}]\!]) = \mathcal{cl}([\![\mathbf{f}]\!]) \Rightarrow \mathsf{KAC} \vdash e = f$.

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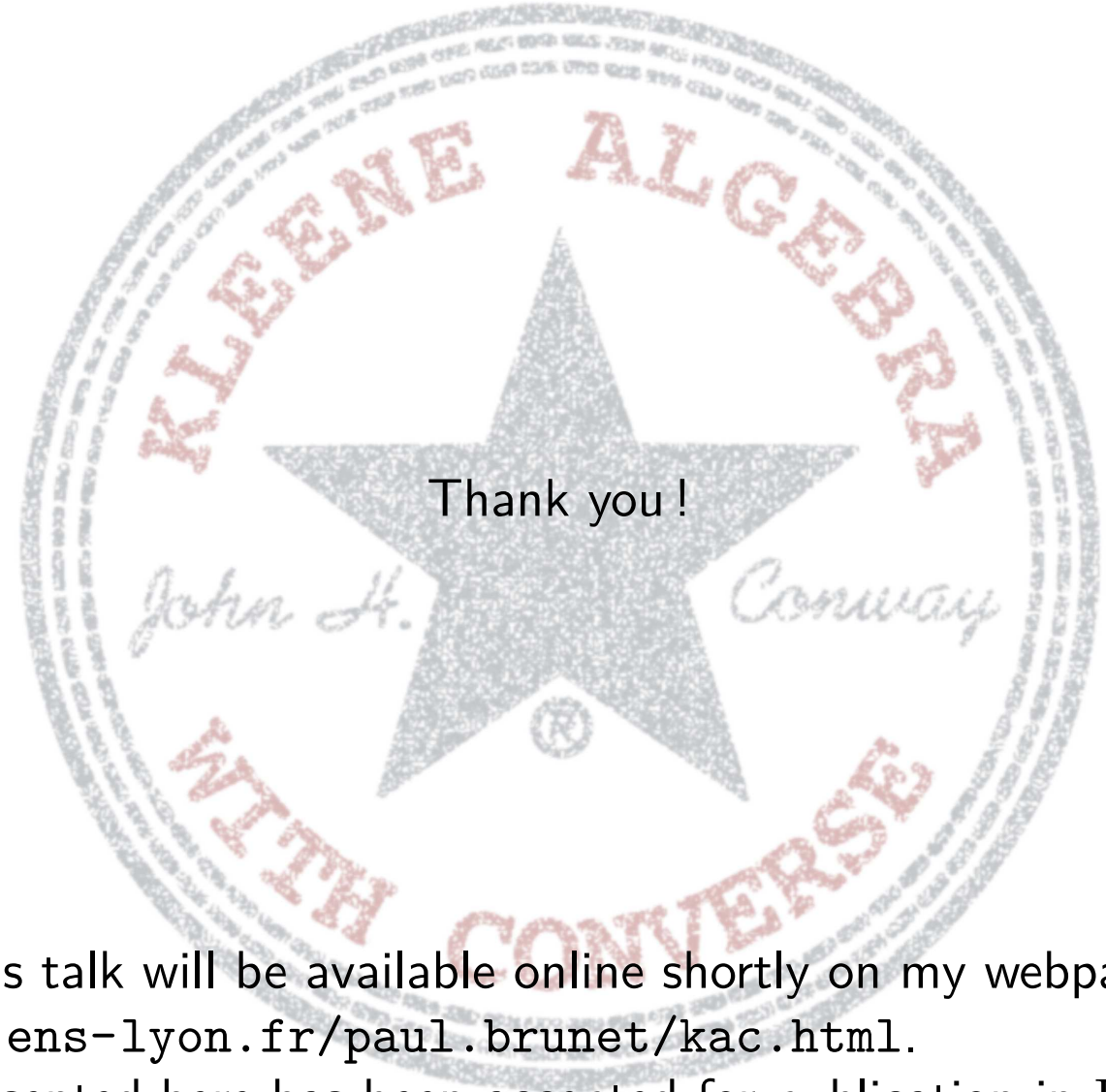
Further work

- Simpler proof of $\mathcal{cl}([\![\mathbf{e}]\!]) = \mathcal{cl}([\![\mathbf{f}]\!]) \Rightarrow \mathsf{KAC} \vdash e = f$.
- Formalize in Coq.
- Other extensions of Kleene Algebra : Action Algebra ($\multimap$), Kleene Algebra with Intersection ($\wedge$)...

That's it !



Thank you !

The slides of this talk will be available online shortly on my webpage `http://perso.ens-lyon.fr/paul.brunet/kac.html`.
The content presented here has been accepted for publication in RAMICS 2014.

Plan