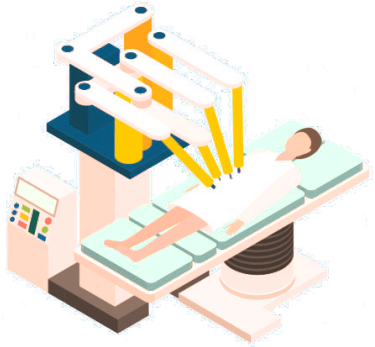
The Equational Theory of Positive Relation Algebra

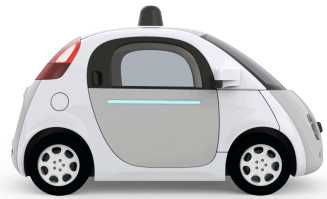
Séminaire MOVE

March 2nd, 2017

Paul Brunet
University College London

Critical software





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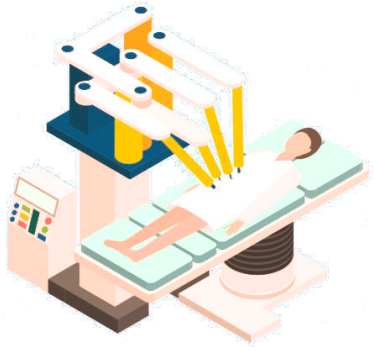

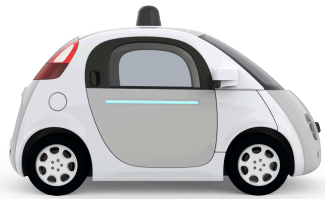

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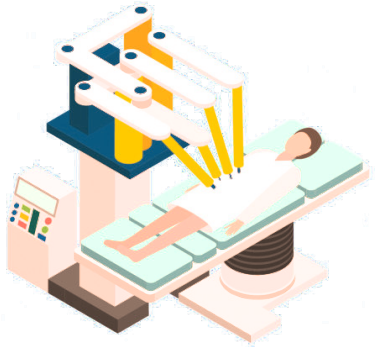

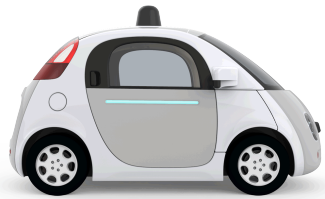

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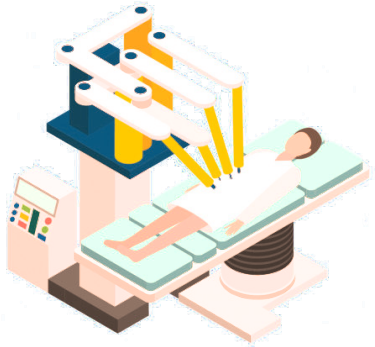
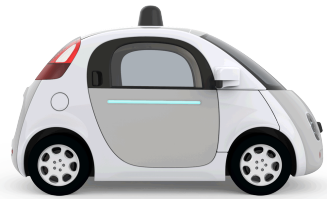
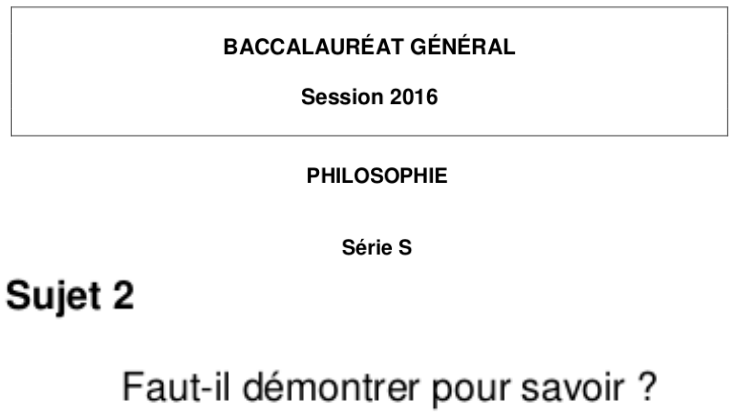




Proofs of correctness

Last year's Baccalauréat's philosophy exam:



BACCALAURÉAT GÉNÉRAL

Session 2016

PHILOSOPHIE

Série S

Sujet 2

Faut-il démontrer pour savoir ?

$\sim$ To know something, does one need a proof of it?

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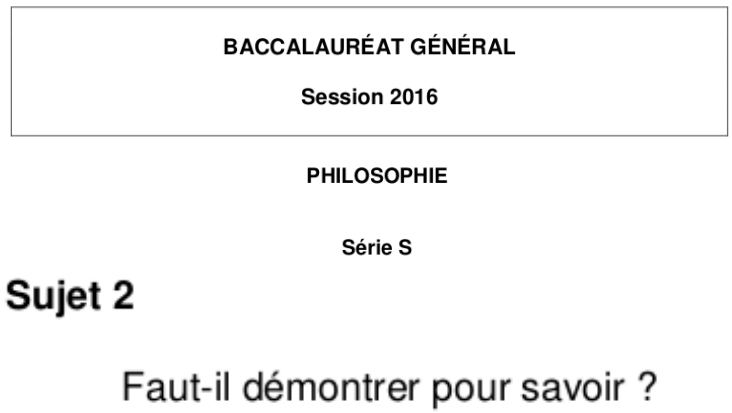

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Yes!

Imperative programs

List of instructions:

```
cmd1;
cmd2;
cmd3;
```

Relation between memory states:

Imperative programs

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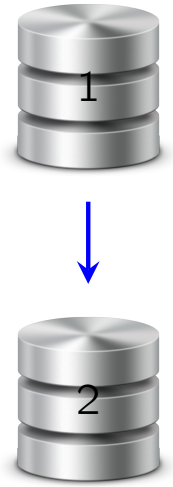

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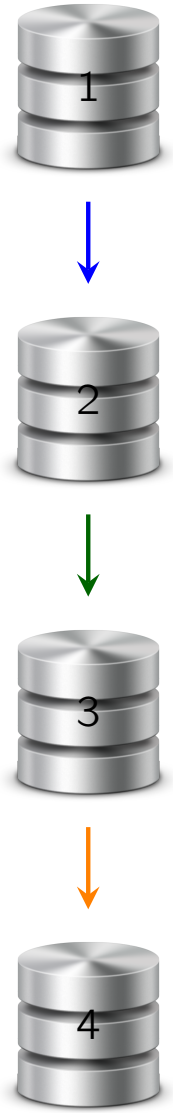

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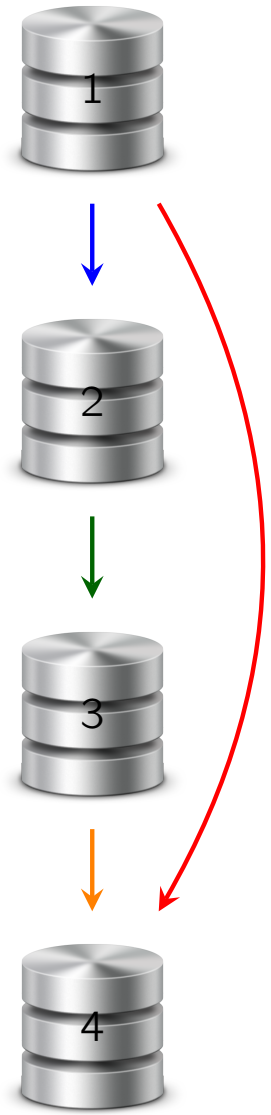

Relational program semantics

$x \leftarrow 1; ((y \leftarrow x) \oplus (y \leftarrow 0))$

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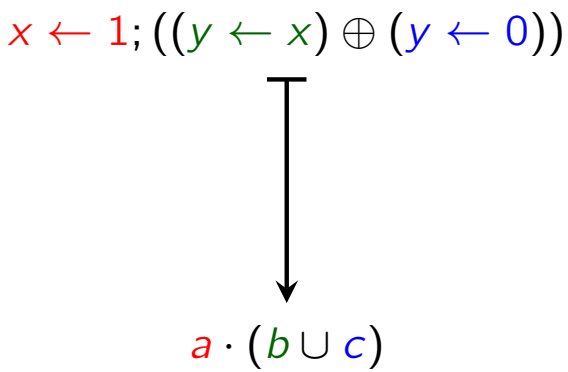

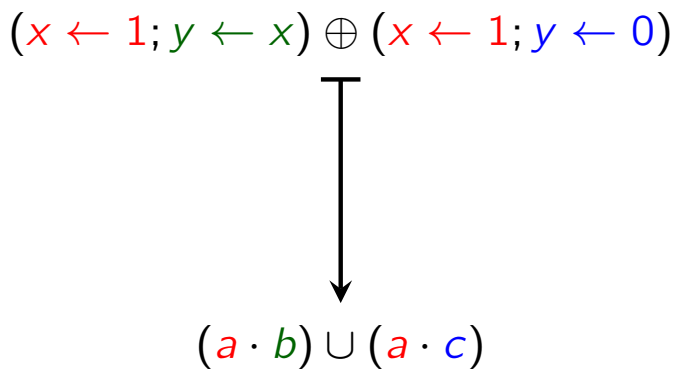

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Relation Algebra

Relational Operators

identity relation	1
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universal relation	$\top$
composition	$R \cdot S$
union	$R \cup S$
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converse	$R^{\smile}$
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complement	R^{c}	$:= \{\langle x, y\rangle \mid \text{not } x\, R\, y\}$

Universal laws of relation algebra

Let O be a set, and R, S and T three binary relations over O.

$$\mathrm{Rel} \models 1 \cup R^\star \cdot S \subseteq (R \cup S)^\star$$

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Simple and boring

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Simple and boring : **could we have a computer do it?**

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↪ **No!** (Undecidable)

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↪ Positive Relation Algebra

Outline

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Regular expressions & languages

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Regular expressions

$$e, f \in \mathrm{Reg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^\star$$

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Regular expressions

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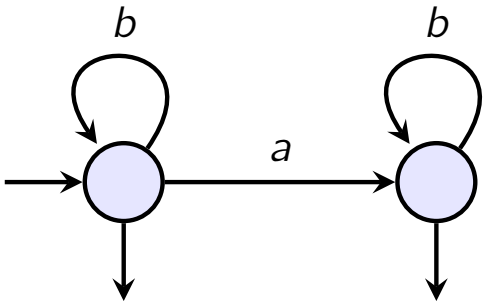
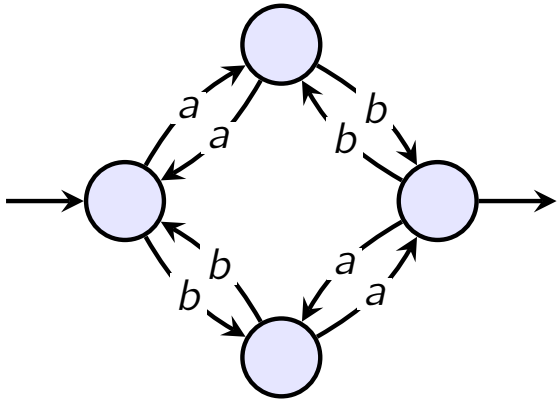
Examples

$$\mathcal{L}(a \cdot (b \cup c)) = \{ab, ac\} \qquad \mathcal{L}(a \cdot 0) = \emptyset$$

$$\mathcal{L}(a^\star) = \{\varepsilon, a, aa, \dots\} = \{a^n \mid n \in \mathbb{N}\}$$

Automata

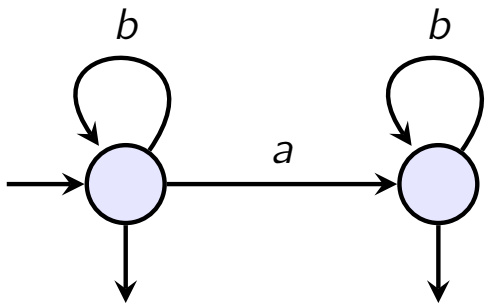


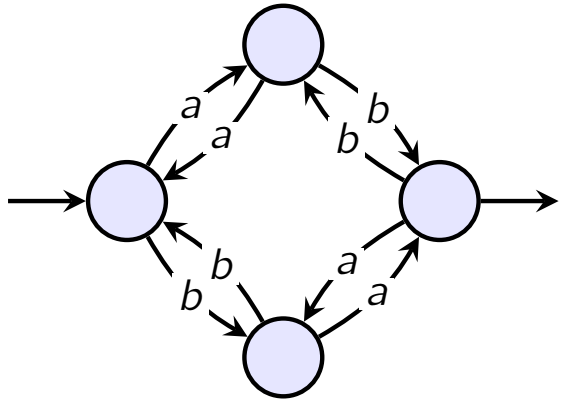


Automata

Theorem

Automata equivalence is decidable.





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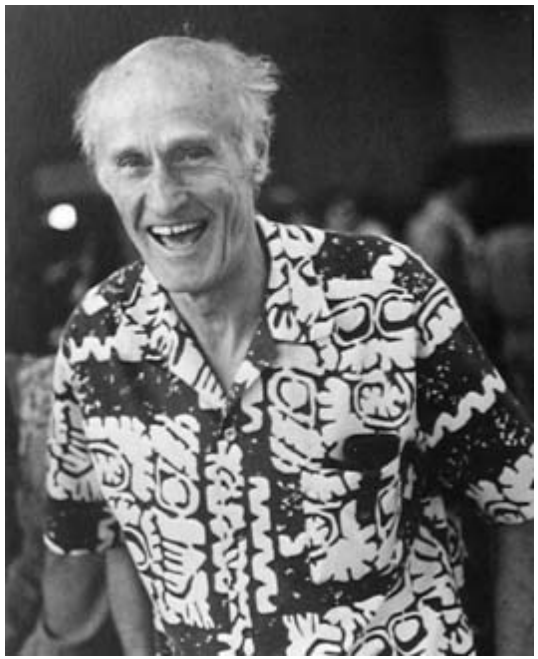

Stephen Cole Kleene

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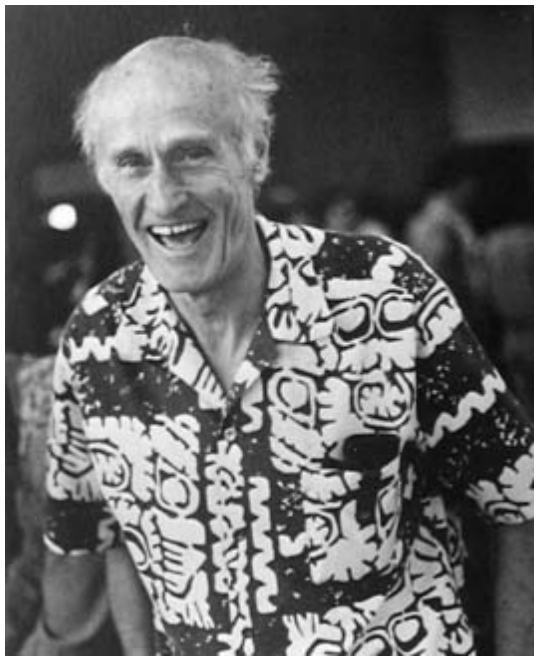
Corollary

There is an algorithm testing if two regular expressions e and f represent the same language.



Stephen Cole Kleene

Kleene Algebra

A Kleene algebra is an algebraic structure $\langle K, \cup, \cdot, ^\star, 0, 1\rangle$ such that

- $\langle K, \cup, \cdot, 0, 1\rangle$ is an idempotent semiring.
- The star satisfies the following laws:

$$1 \cup a \cdot a^\star \leqslant a^\star$$

$$b \cup a \cdot x \leqslant x \Rightarrow a^\star \cdot b \leqslant x$$

(+ symmetric)

John Horton Conway

Examples:

Algebras of languages: the set of languages over some finite alphabet A.

Algebras of relations: the set of binary relations over some set O.

Equivalence of expressions

Everything is equivalent

$$\forall O, \forall a, b, c, ... \in \mathcal{P}\left(O \times O\right), e = f ?$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.

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$$\mathrm{Rel} \models e = f \Longleftrightarrow \mathcal{L}\left(e\right) = \mathcal{L}\left(f\right)$$

Equivalence of expressions

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$$\forall A, \forall a, b, c, ... \in \mathcal{P}\left(A^{\star}\right), e = f ?$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.
- $\mathrm{Lang} \models e = f$: universal law of language Kleene algebra.

$$\mathrm{Rel} \models e = f \Longleftrightarrow \mathcal{L}(e) = \mathcal{L}(f) \Longleftrightarrow \mathrm{Lang} \models e = f$$

Equivalence of expressions

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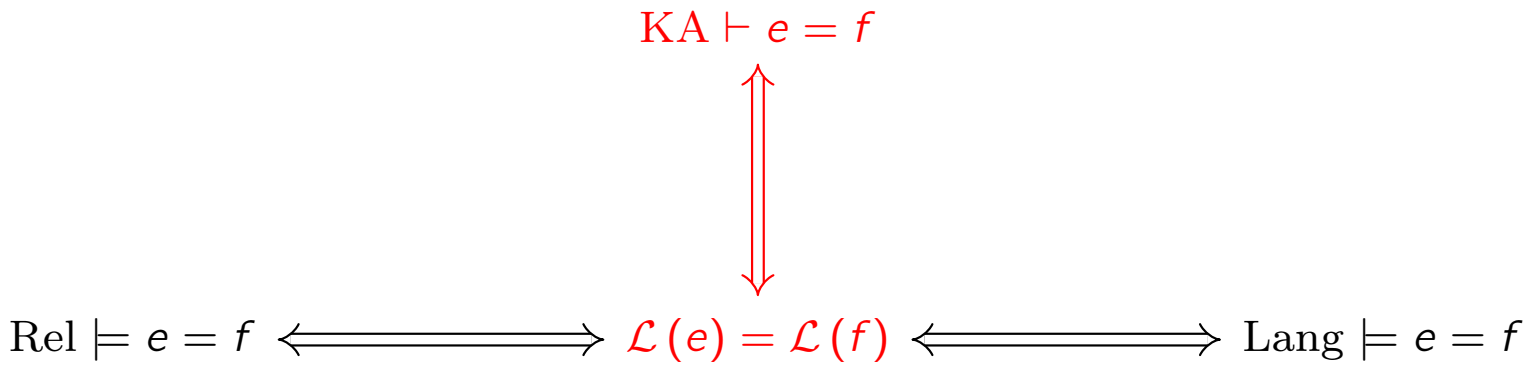
$$\forall K, \forall a, b, c, ... \in K, e = f ?$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.
- $\mathrm{Lang} \models e = f$: universal law of language Kleene algebra.
- $\mathrm{KA} \vdash e = f$: universal law of Kleene algebra.
  (i.e. logical consequence of the axioms of KA).



Equivalence of expressions

Everything is equivalent, and decidable

$$\forall K, \forall a, b, c, ... \in K, e = f ?$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.
- $\mathrm{Lang} \models e = f$: universal law of language Kleene algebra.
- $\mathrm{KA} \vdash e = f$: universal law of Kleene algebra.
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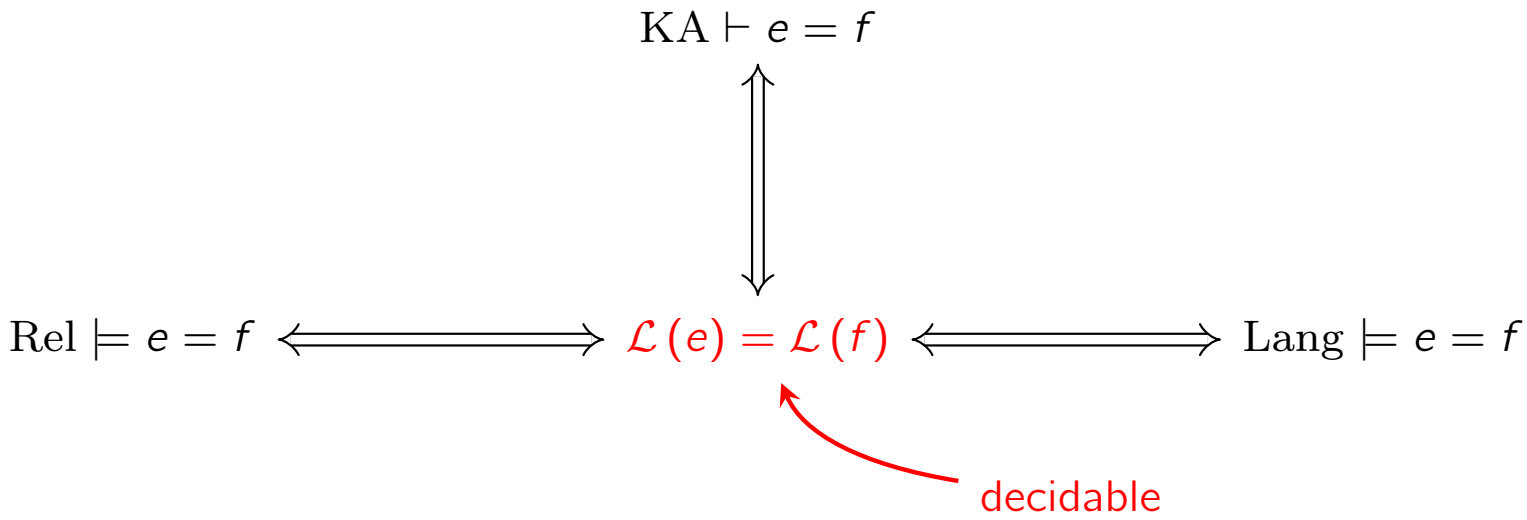

Extensions of Kleene Algebra

Relational Operators

identity relation	:	1	✓
empty relation	:	0	✓
universal relation	:	$\top$	
composition	:	$R \cdot S$	✓
union	:	$R \cup S$	✓
intersection	:	$R \cap S$	
converse	:	$R^\smile$	
refl. trans. closure	:	$R^\star$	✓
complement	:	R^c	

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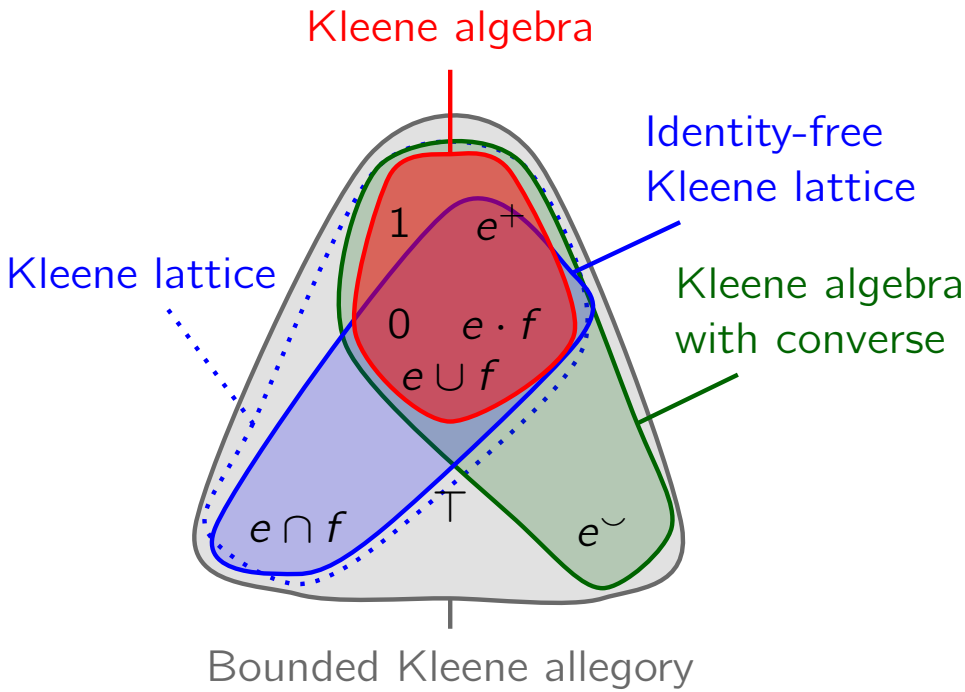
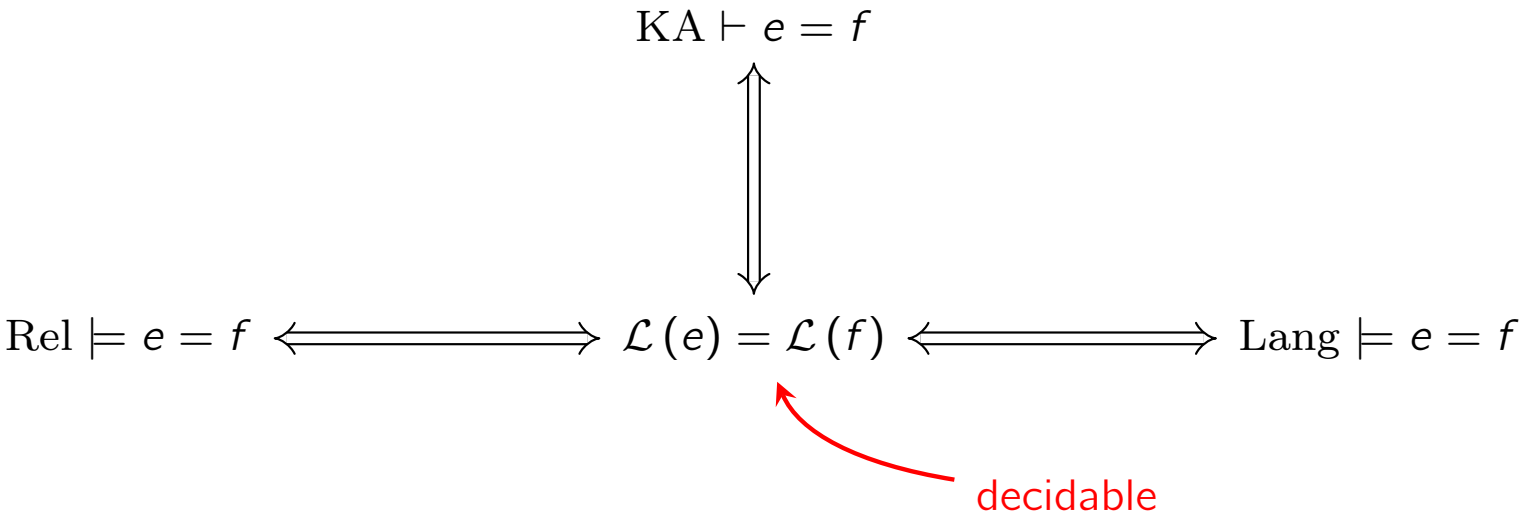


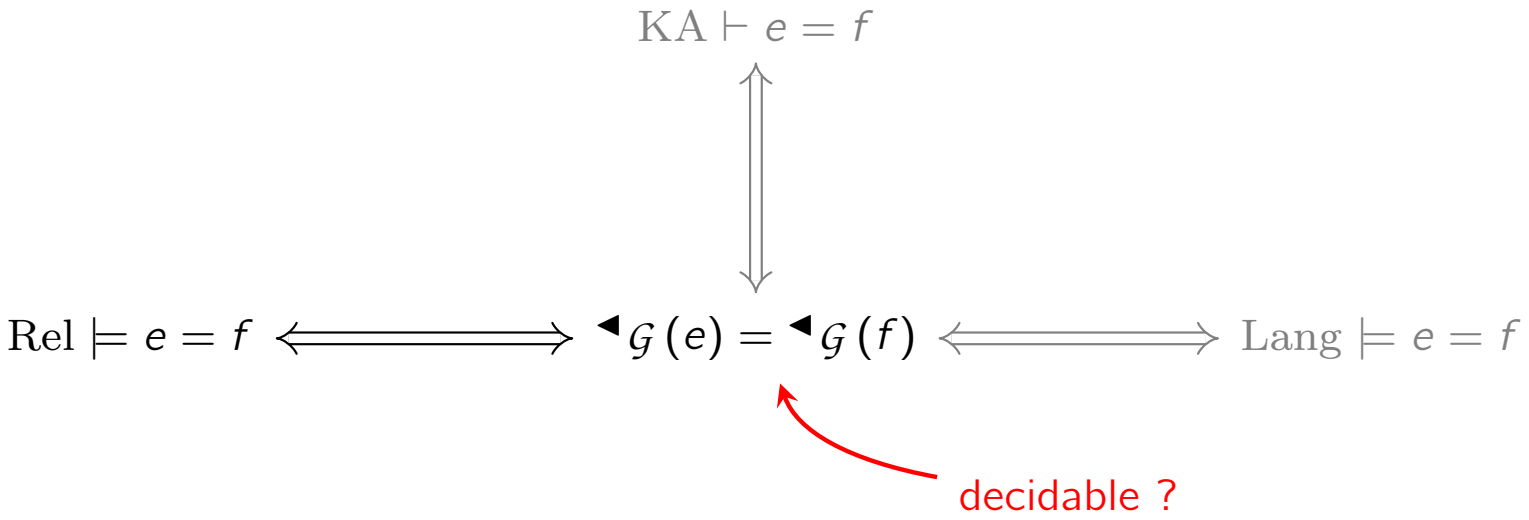
Equivalence of expressions

Regular expressions e, f.



Equivalence of expressions

Richer expressions e, f.



Outline

Kleene Allegories

$$e, f \in \mathsf{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^\star \mid e^\smile \mid \top$$

Kleene Allegories

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Kleene Allegories

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$$\mathrm{Rel} \models e = f \not\Leftarrow \mathcal{L}(e) = \mathcal{L}(f) \qquad \mathrm{Rel} \models e = f \not\Rightarrow \mathcal{L}(e) = \mathcal{L}(f)$$

Counterexample

$\mathcal{L}(a \cap b) = \mathcal{L}(0)$	$\mathcal{L}(a) = \mathcal{L}(a^\smile)$	$\mathcal{L}(\top a \top b \top) \neq \mathcal{L}(\top b \top a \top)$
$\mathrm{Rel} \not\models a \cap b = 0$	$\mathrm{Rel} \not\models a = a^\smile$	$\mathrm{Rel} \models \top a \top b \top = \top b \top a \top$

A different approach is needed.

Graphs/Ground terms

$$u, v \in \mathsf{W}_{\Sigma} ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$

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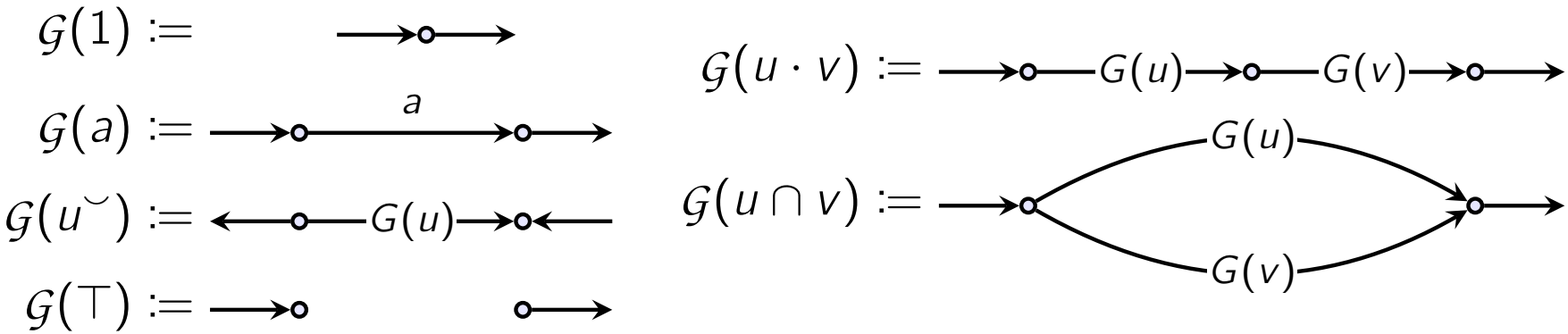
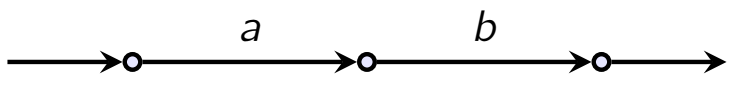


Examples

$\mathcal{G}(a \cdot b)$: 

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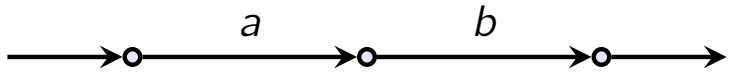
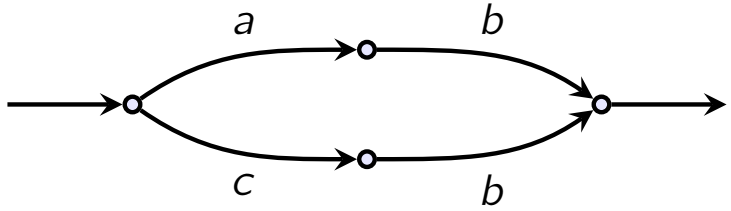


$\mathcal{G}((a \cdot b) \cap (c \cdot b))$:



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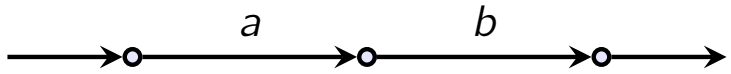

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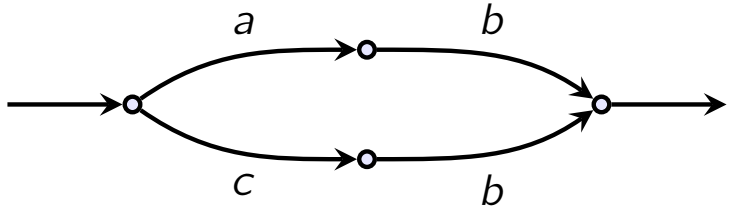
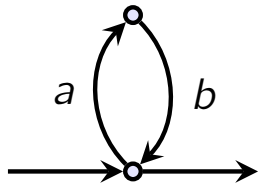
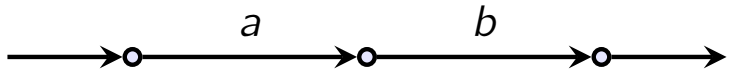


$\mathcal{G}((a \cdot b) \cap 1)$:



Examples

$\mathcal{G}(a \cdot b)$:



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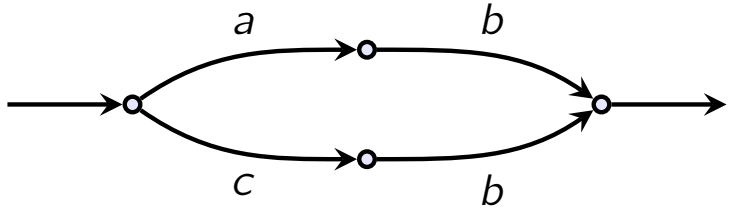
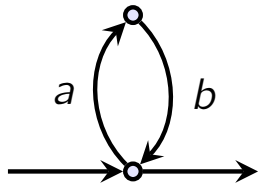
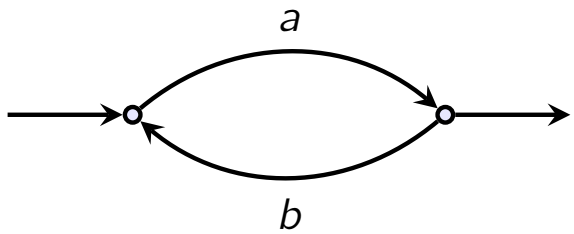


$\mathcal{G}((a \cdot b) \cap 1)$:

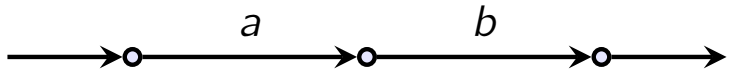


$\mathcal{G}(a \cap b^{\smile})$:



Examples

$\mathcal{G}(a \cdot b)$:



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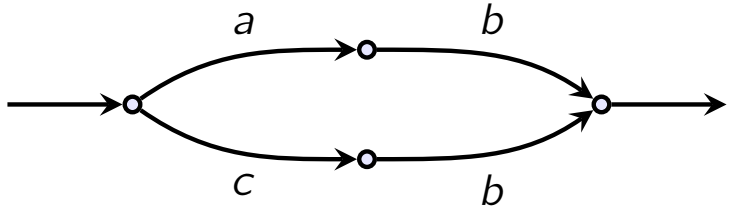
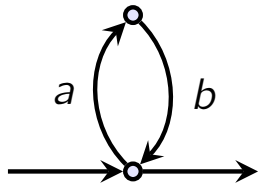


$\mathcal{G}((a \cdot b) \cap 1)$:



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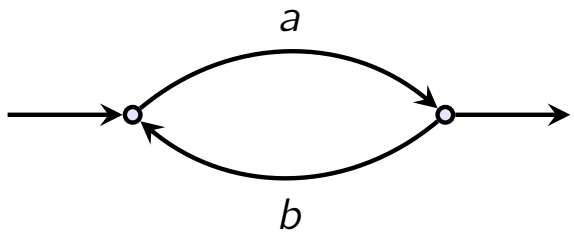


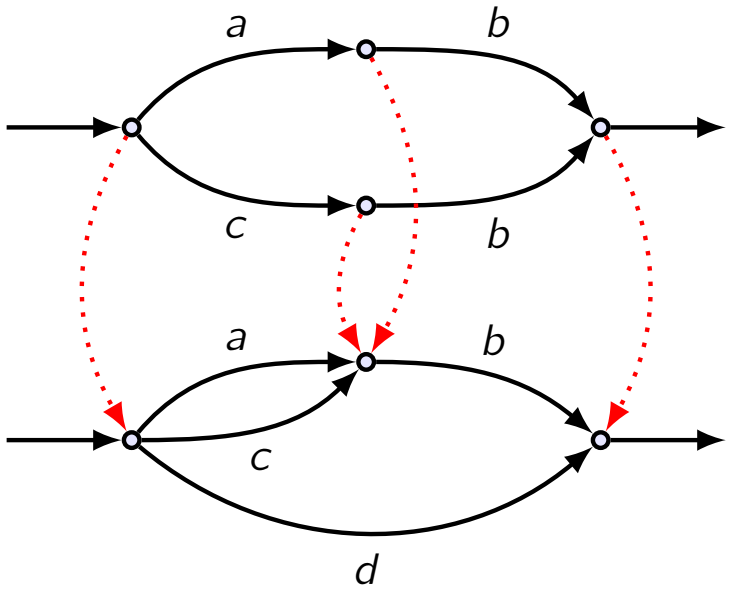
$\mathcal{G}(\top a \top b \top)$:

Preorder

Preorder on graphs

$G \blacktriangleleft H$ if there exists a graph morphism from H to G.

H :



$((a \cap c) \cdot b) \cap d$

G :

$(a \cdot b) \cap (c \cdot b)$

Characterization theorem

$$u, v \in \mathsf{W}_\Sigma ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^\smile \mid \top$$

Theorem

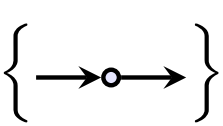
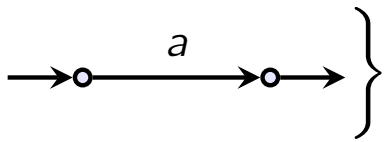
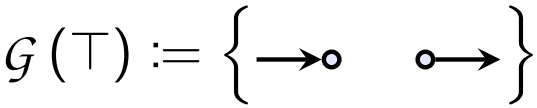
$$\mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v)$$

Freyd & Scedrov, **Categories, Allegories**, *1990*

Andréka & Bredikhin, **The equational theory of union-free algebras of relations**, *1995*

Graph languages

$$e, f \in \mathsf{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^\star \mid e^\smile \mid \top$$

$\mathcal{G}(1) :=$  $\mathcal{G}(a) :=$  $\mathcal{G}(\top) :=$ 

$$\mathcal{G}\left(e^\smile\right) := \left\{G^\smile \mid G \in \mathcal{G}(e)\right\}$$

$$\mathcal{G}(e \cdot f) := \{G \cdot G' \mid G \in \mathcal{G}(e) \text{ and } G' \in \mathcal{G}(f)\}$$

$$\mathcal{G}(e \cap f) := \{G \cap G' \mid G \in \mathcal{G}(e) \text{ and } G' \in \mathcal{G}(f)\}$$

$$\mathcal{G}(0) := \emptyset \qquad \mathcal{G}(e \cup f) := \mathcal{G}(e) \cup \mathcal{G}(f) \qquad \mathcal{G}(e^\star) := \bigcup_{n \in \mathbb{N}} \mathcal{G}(e)^n.$$

Characterization theorem

${}^{\blacktriangleleft}S$ is the downwards closure of S with respect to $\blacktriangleleft$.
${}^{\blacktriangleleft}S \coloneqq \{G \mid \exists H \in S : G \blacktriangleleft H\}$.

Theorem

$e, f \in \mathsf{AReg}\langle\Sigma\rangle$,

$$\mathrm{Rel} \models e \subseteq f \Leftrightarrow {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f)$$

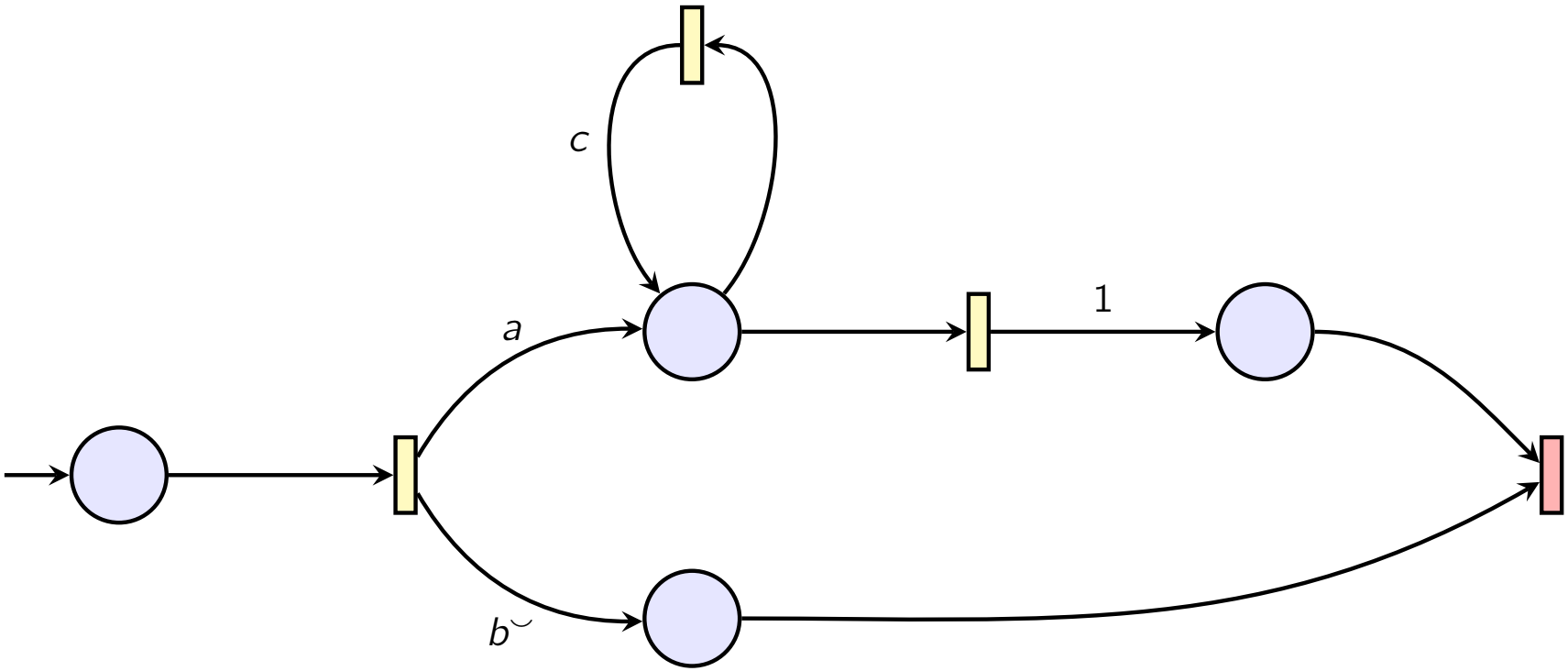
B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Follows easily from:

Andréka, Mikulás & Németi, **The equational theory of Kleene lattices**, *2011*

Outline

Petri automata



Set of labels: $\Sigma' \coloneqq \Sigma \cup \left\{ a^\smile \mid a \in \Sigma \right\} \cup \{1, \top\}$.

Language generated vs. Language recognised

Let $\mathcal{A}$ be a finite state word automaton.

The language of $\mathcal{A}$ is:

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The language of $\mathcal{A}$ is:

- the set of words **accepted** by $\mathcal{A}$; **recognised**

Language generated vs. Language recognised

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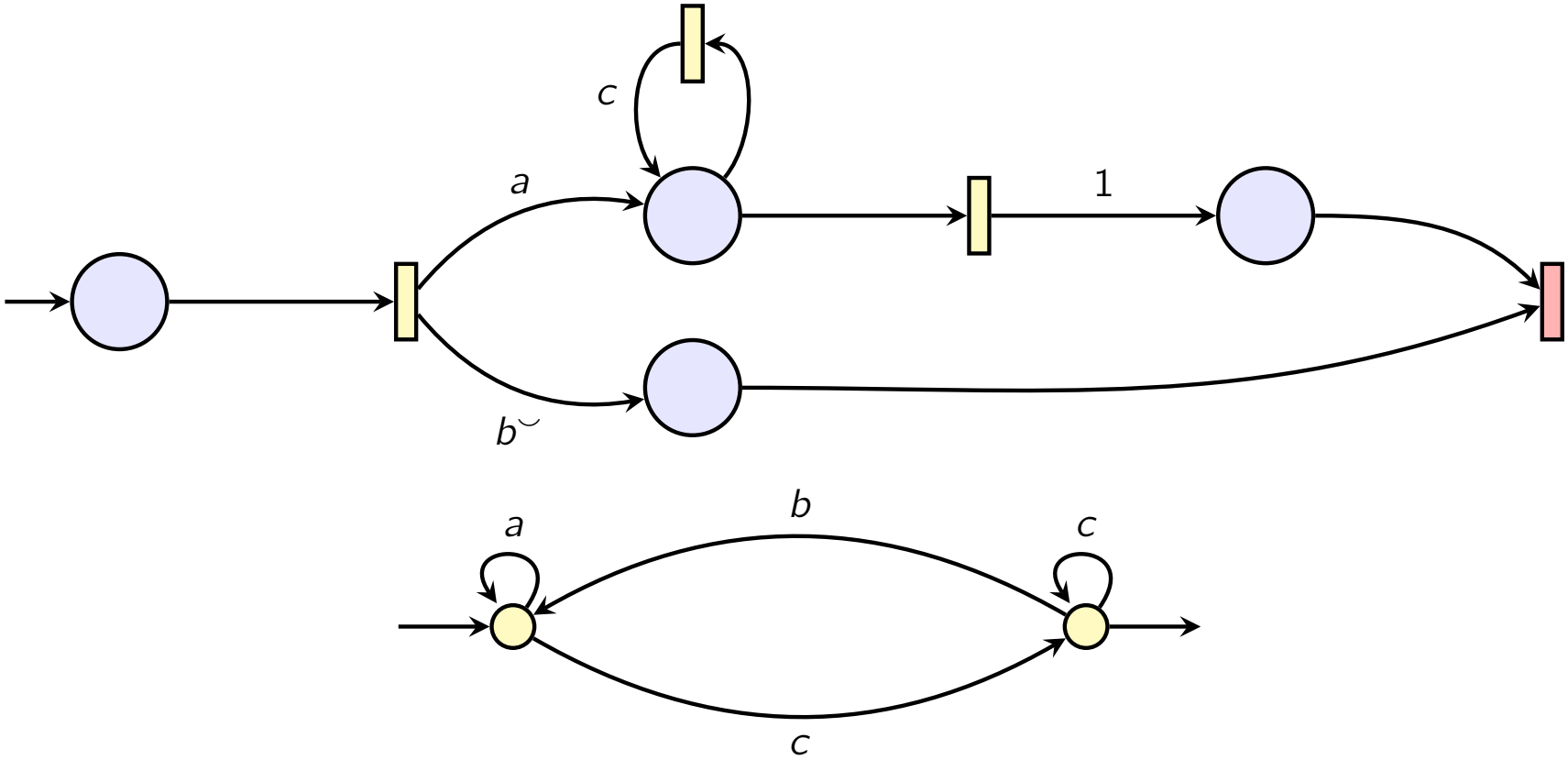
The language of $\mathcal{A}$ is:

- the set of words **accepted** by $\mathcal{A}$; **recognised**
- the set of words labelling **accepting paths** in $\mathcal{A}$. **generated**

Language recognised by an automaton



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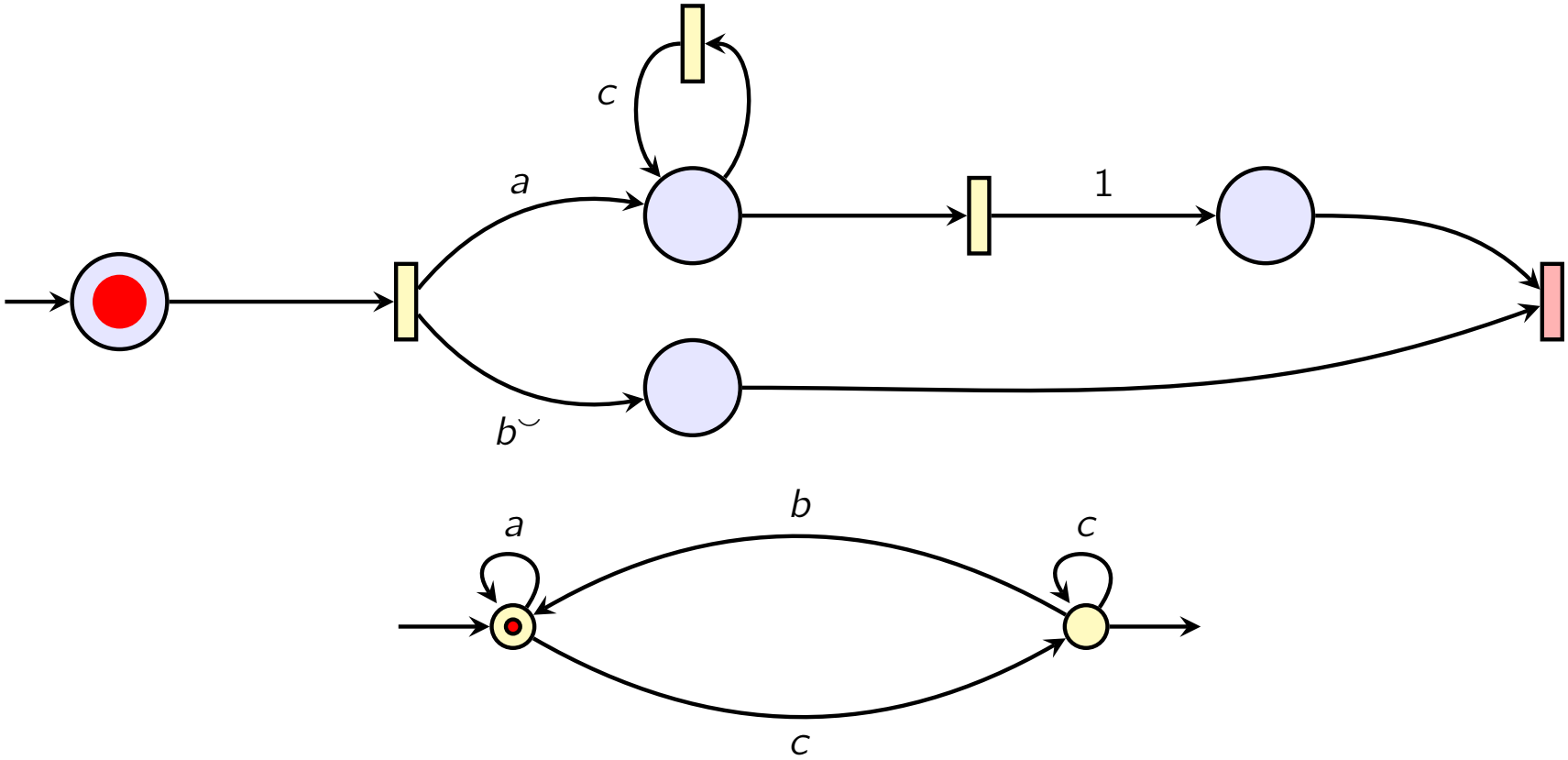

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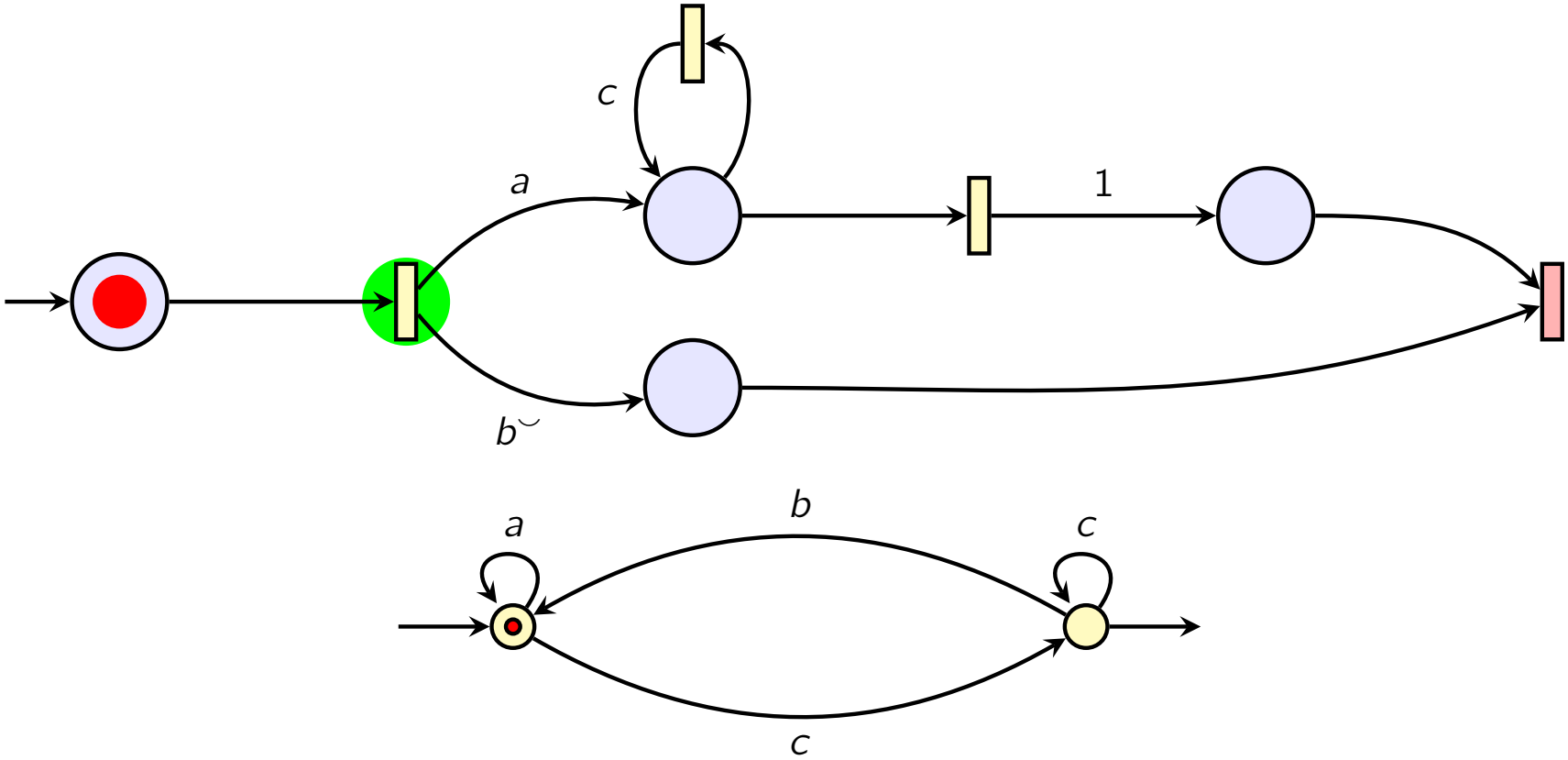

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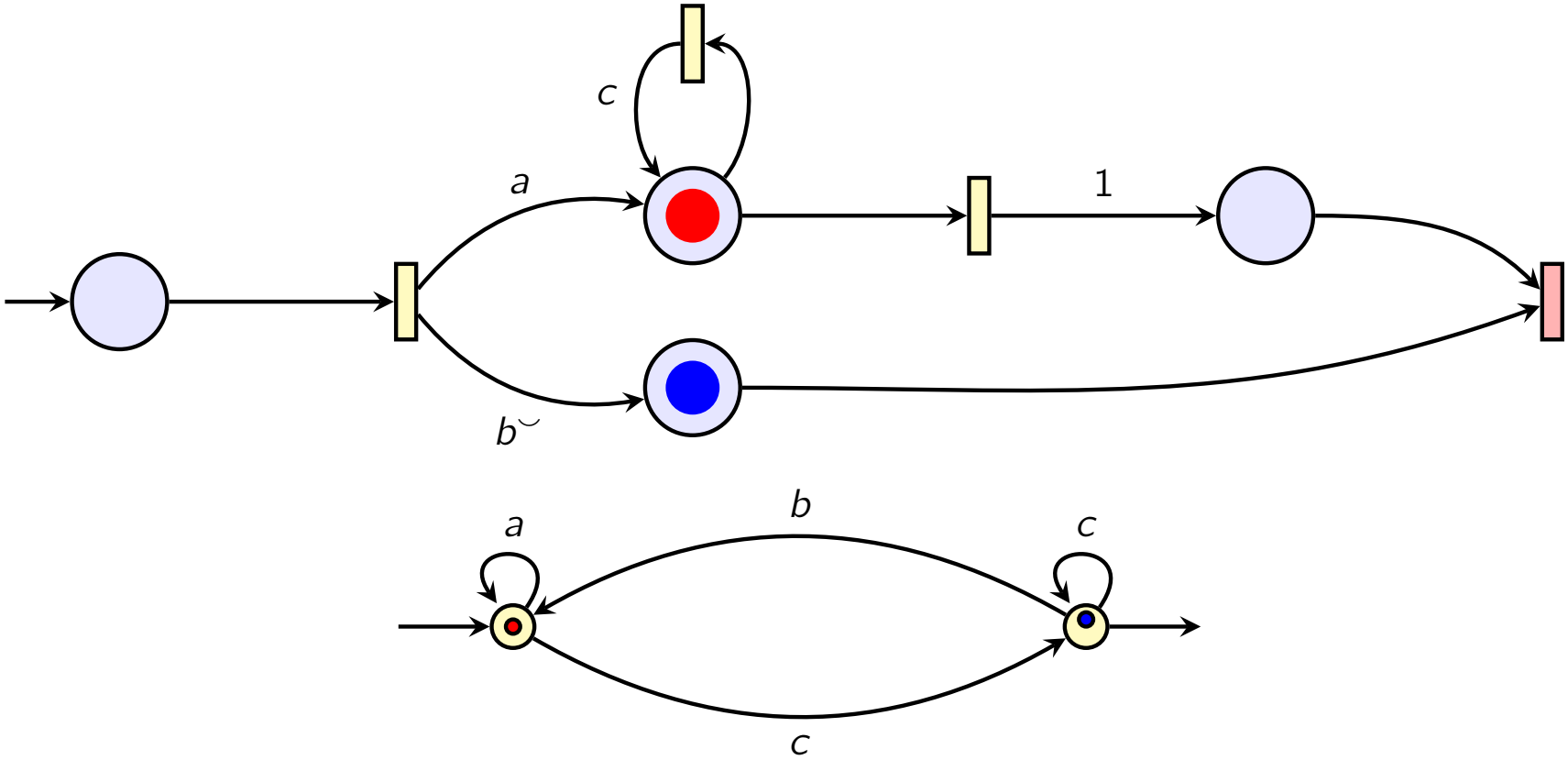

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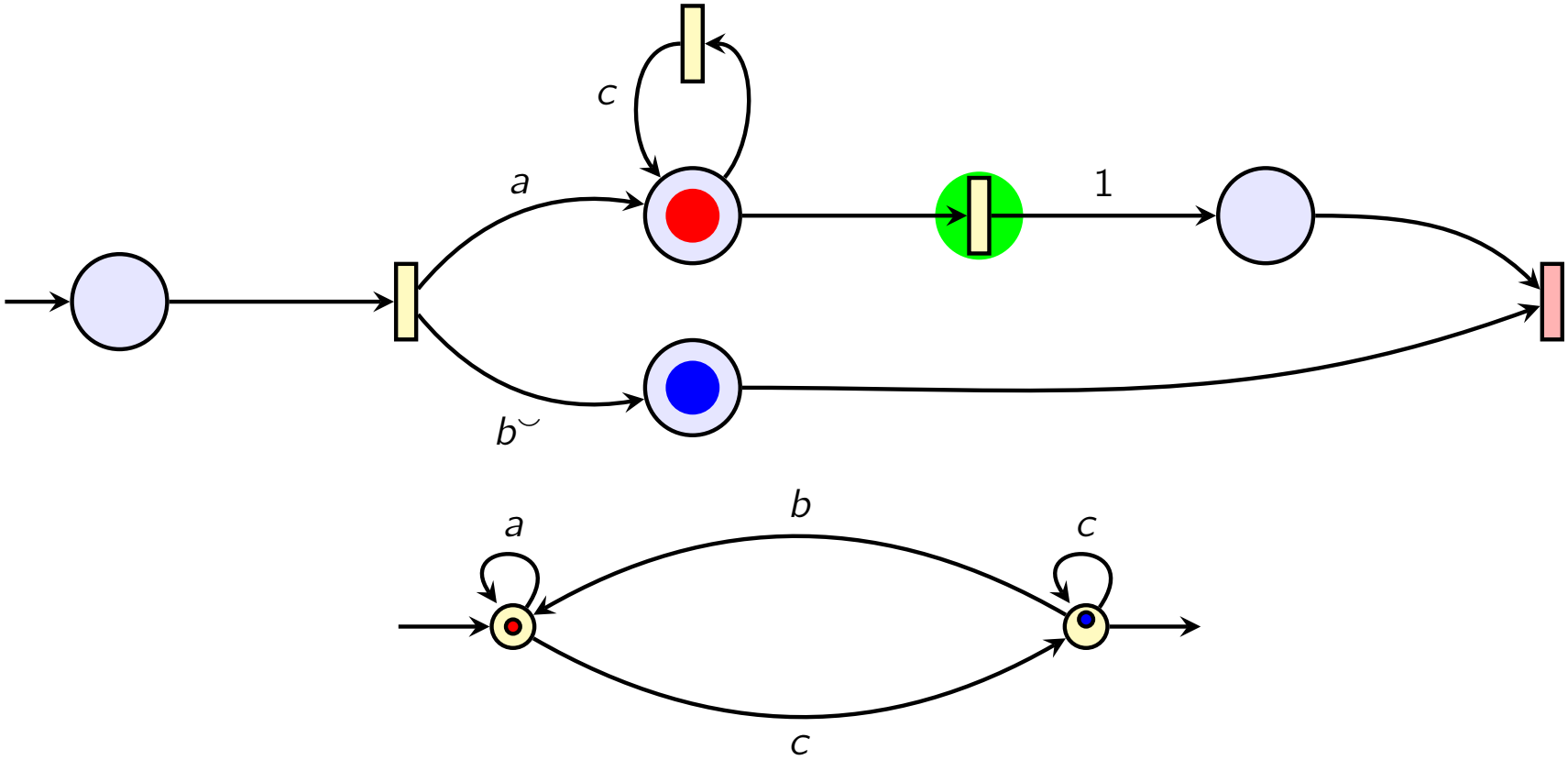

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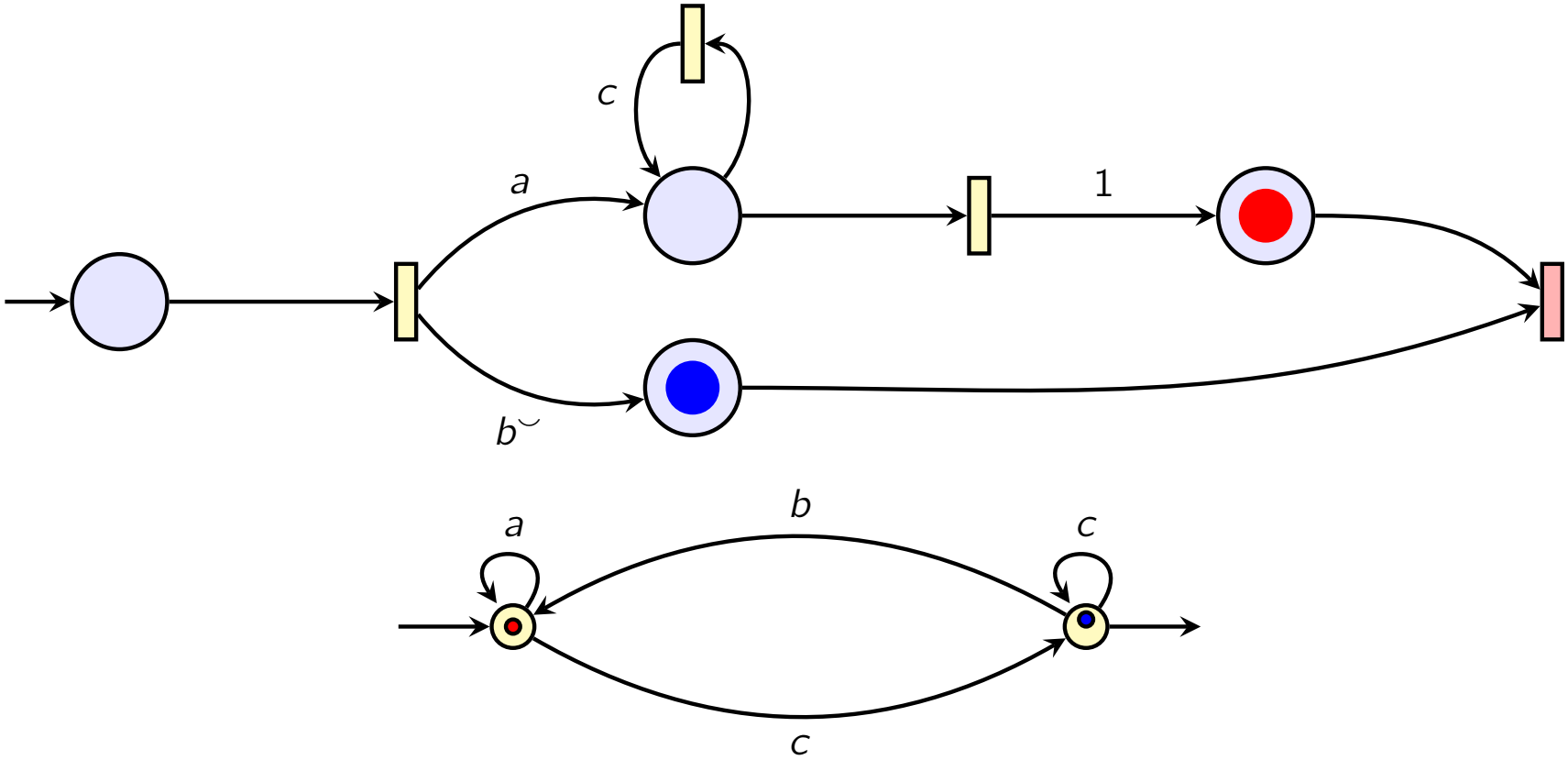

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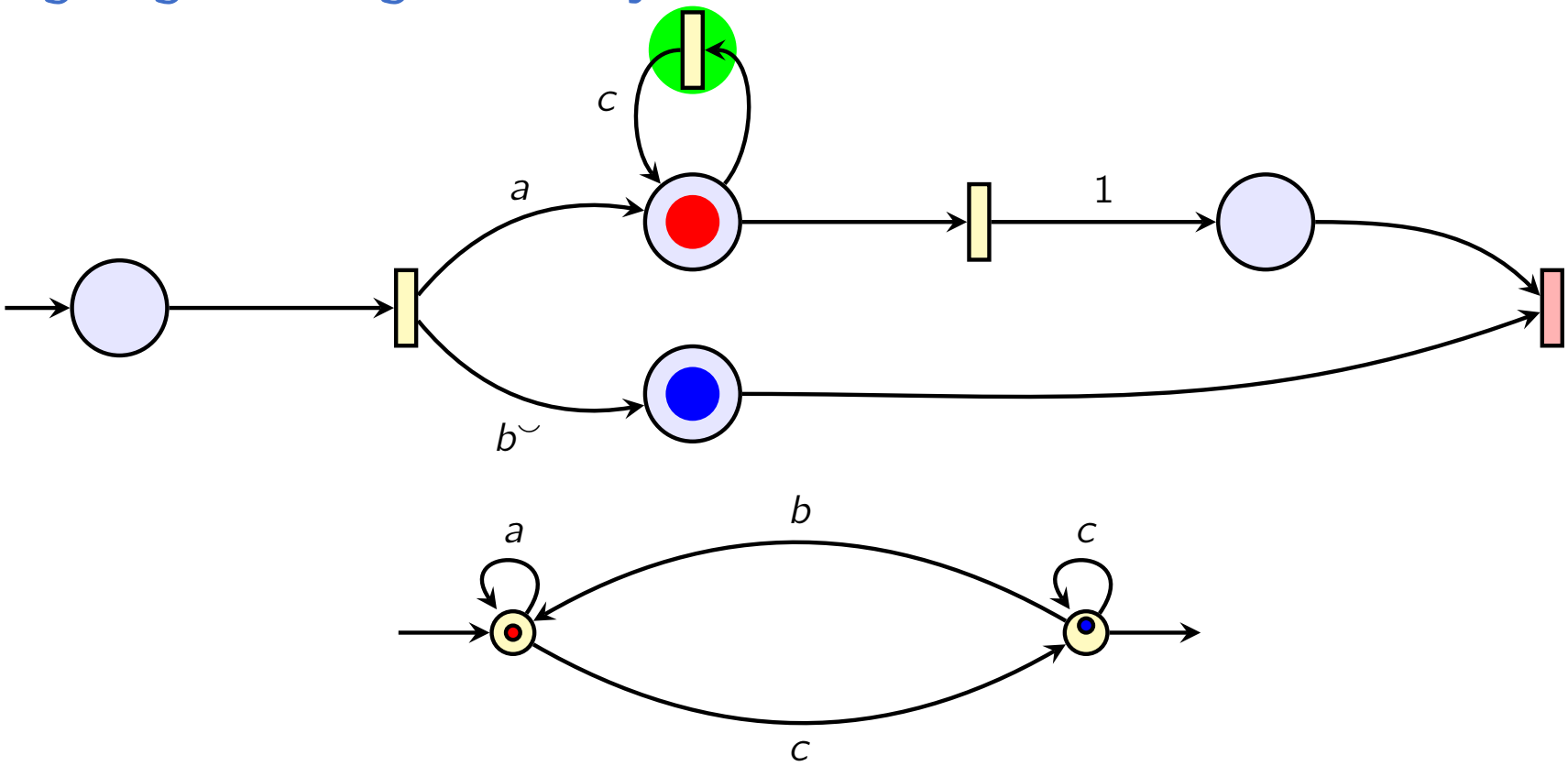

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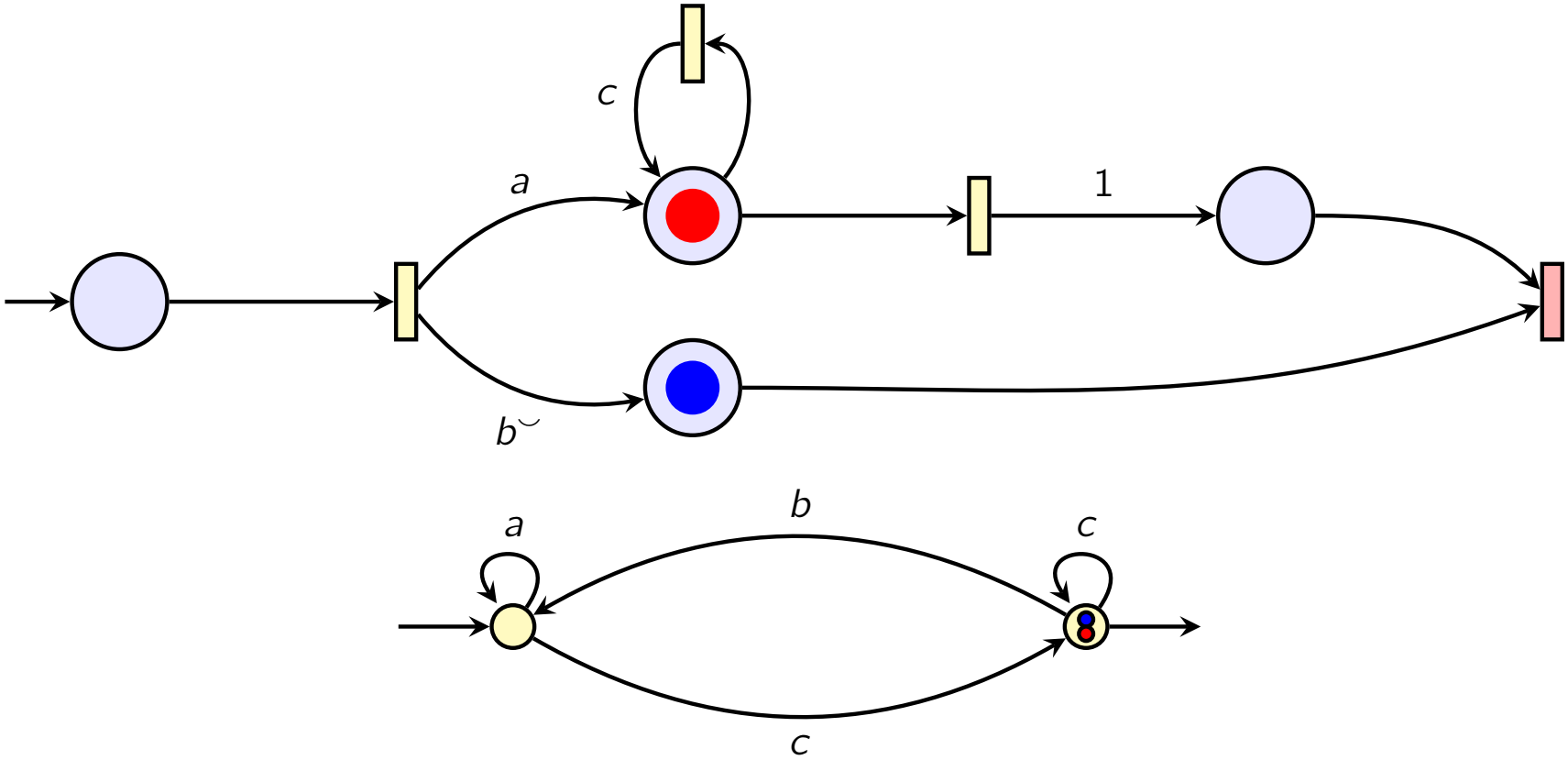

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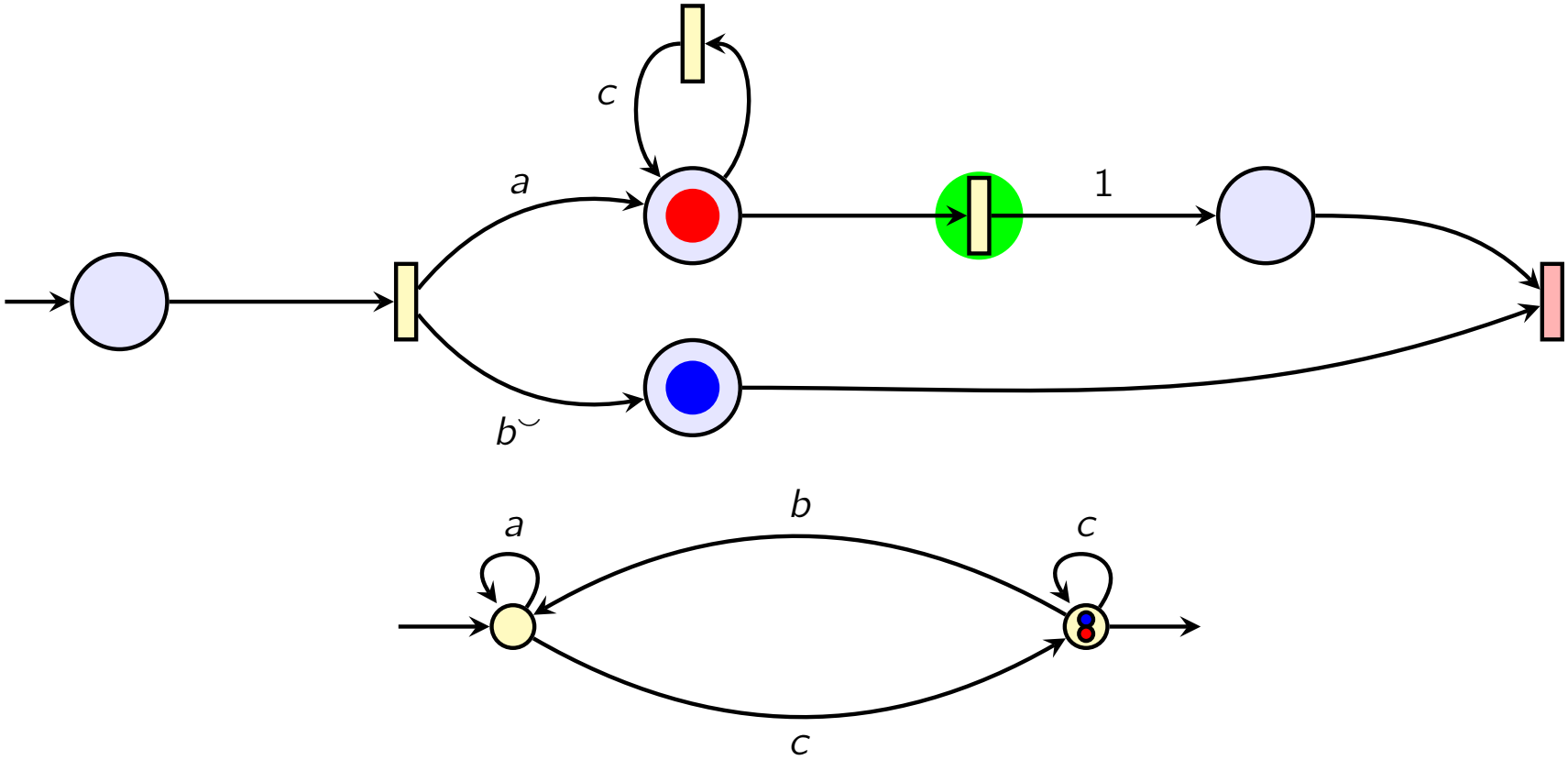

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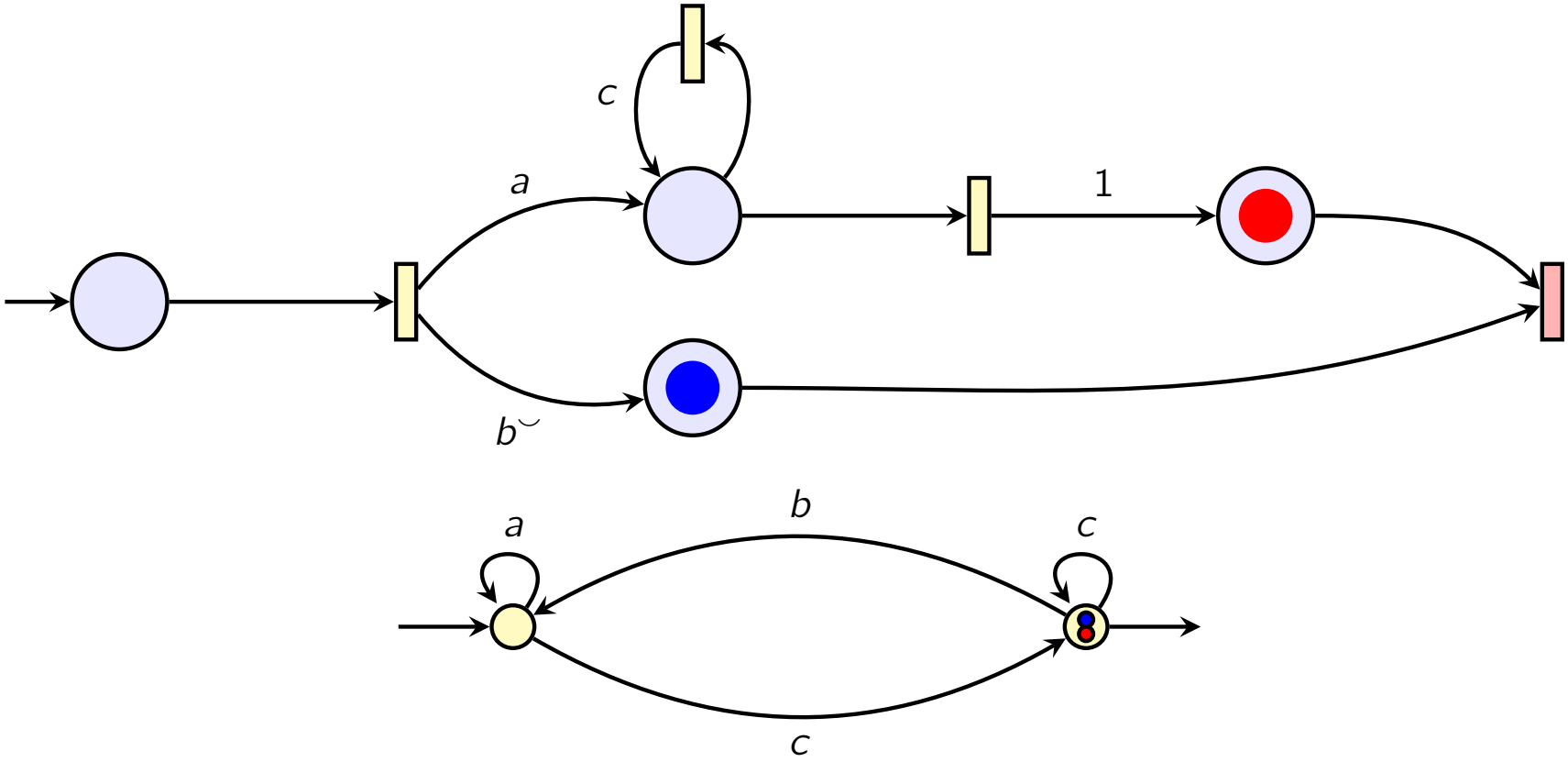

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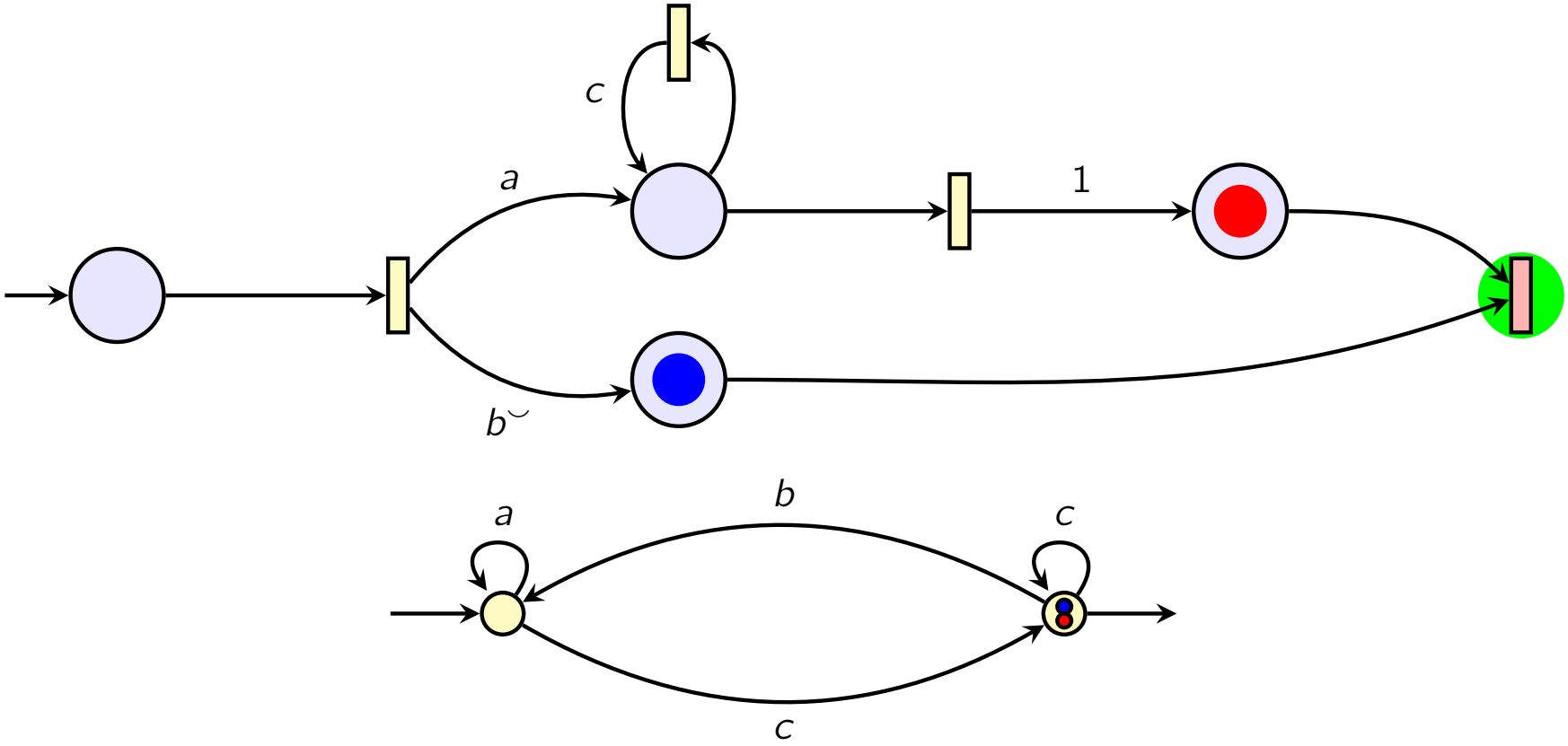

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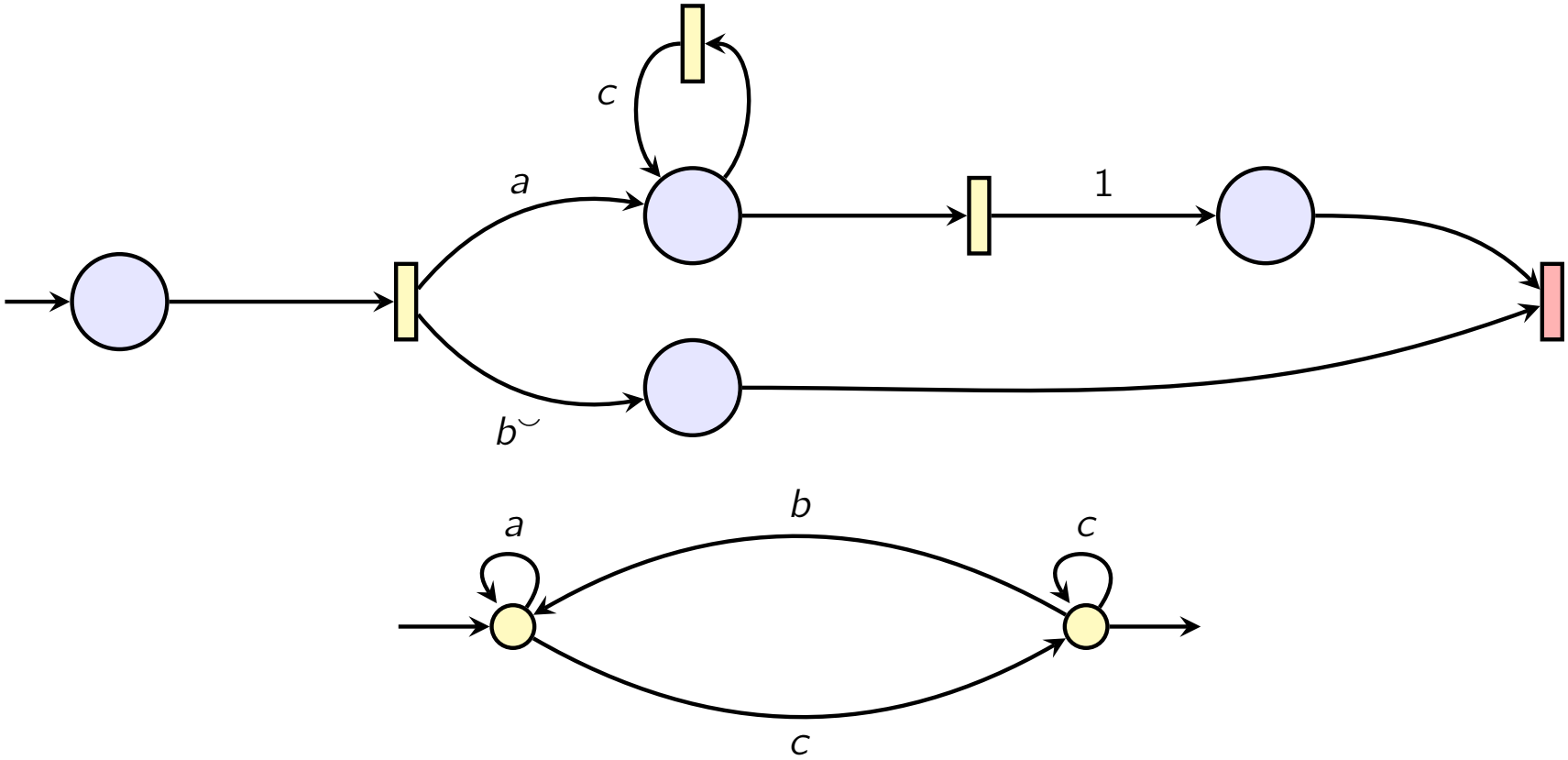
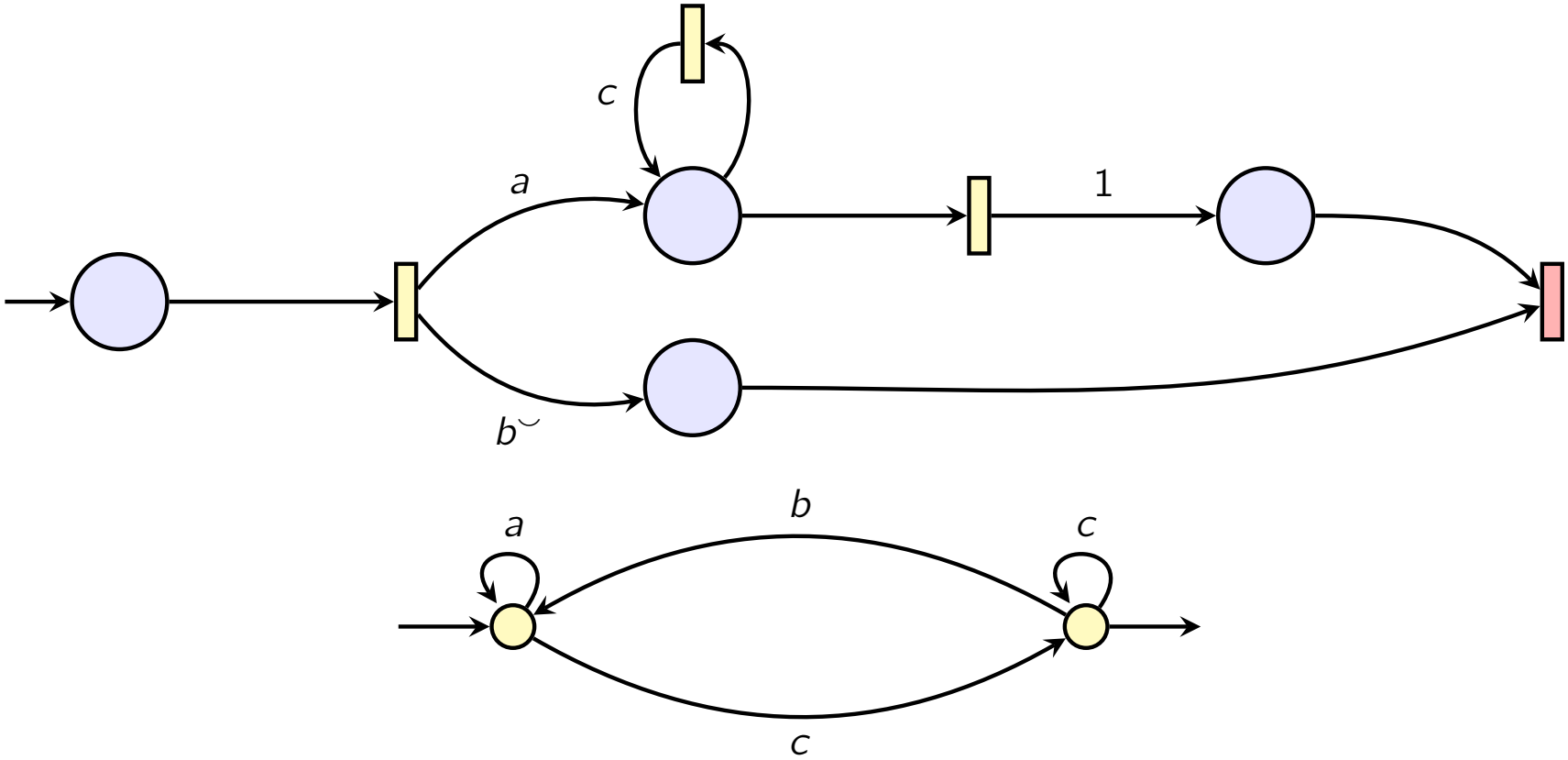


Success!

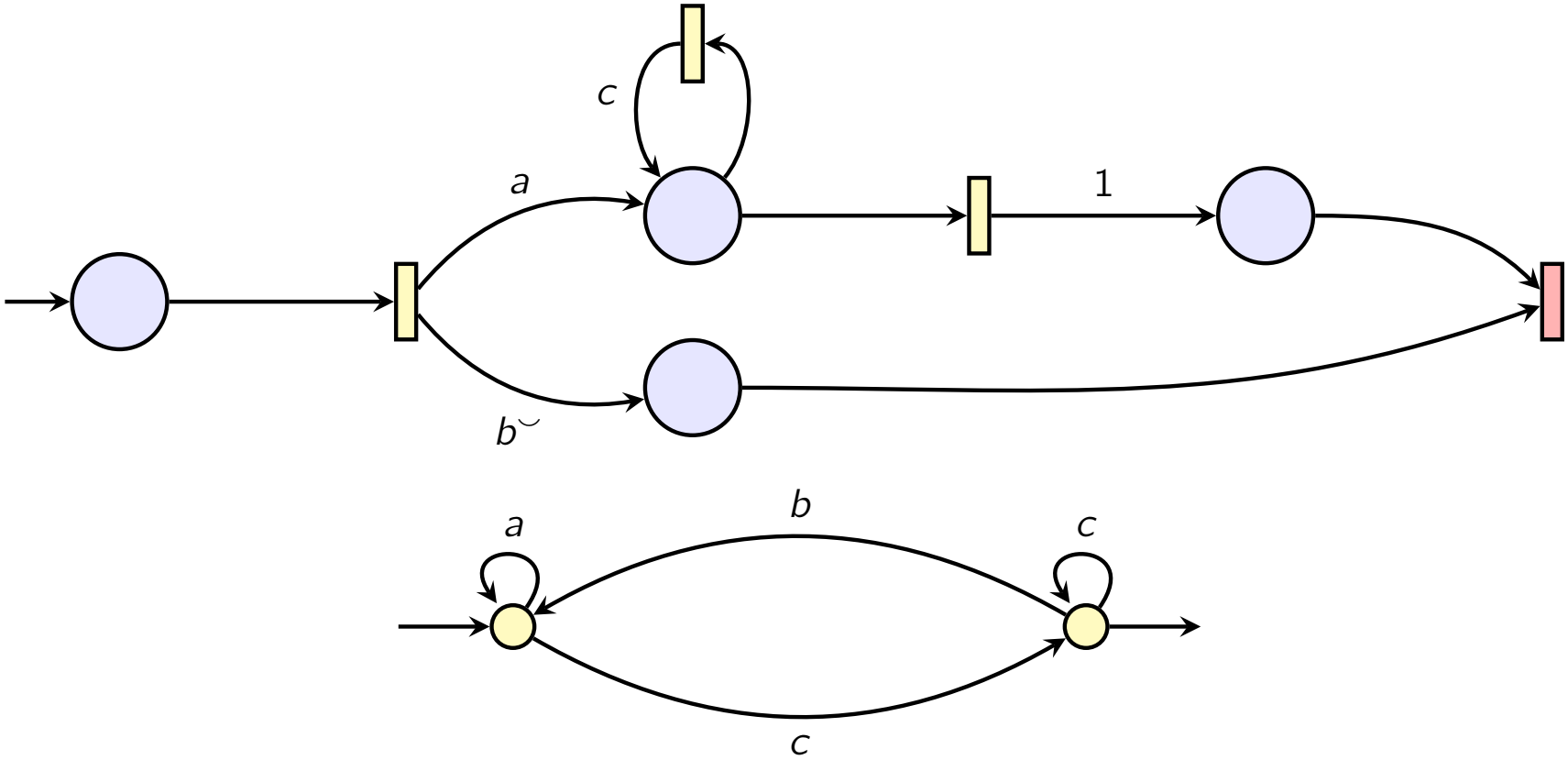
Language recognised by an automaton



Language of $\mathcal{A}$

$$\mathcal{L}(\mathcal{A}) = \{G \mid \mathcal{A} \text{ accepts } G\}$$

Language recognised by an automaton



Language of $\mathcal{A}$

$$\mathcal{L}(\mathcal{A}) = \{G \mid \mathcal{A} \text{ accepts } G\}$$

Here: $\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\mathcal{G}\left((a \cdot c^\star) \cap b^\smile\right)$.

Petri automata for Kleene allegories

Correctness

For any $e \in \mathsf{AReg}\langle\Sigma\rangle$,

$$e$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathsf{AReg}\langle\Sigma\rangle$,

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B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathsf{AReg}\langle\Sigma\rangle$,

$$\mathcal{L}(\mathcal{A}(e))$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

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For any $e \in \mathsf{AReg}\langle\Sigma\rangle$,

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Petri automata for Kleene allegories

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For any $e \in \mathsf{AReg}\langle\Sigma\rangle$,

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B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

So far:

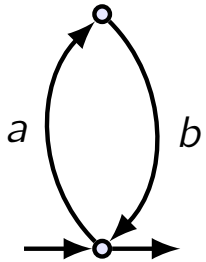
$e, f \in \mathsf{AReg}\langle\Sigma\rangle$

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f) \;\Leftrightarrow\; \mathcal{L}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)).$$

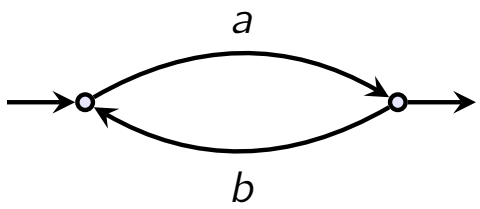
Outline

Restriction: identity-free Kleene lattice terms

$\mathcal{G}((a \cdot b) \cap 1)$:



$\mathcal{G}\left(a \cap b^{\smile}\right)$:



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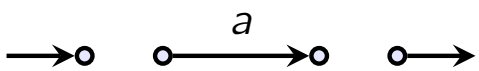

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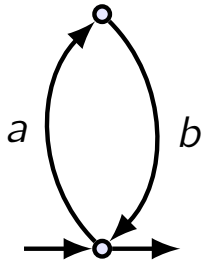

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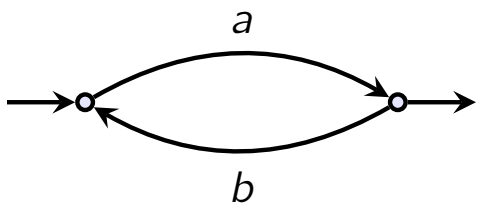

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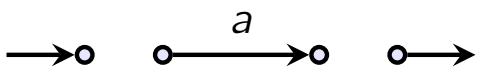

Identity-free Kleene Lattice

$$u, v \in \mathrm{W}_{\Sigma}^{-} ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$

$$e, f \in \mathrm{GReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{+} \mid e^{\smile} \mid \top$$

$\hookrightarrow$ Series-Parallel graphs & simple automata

Decision procedure

$e, f \in \mathsf{GReg}\langle\Sigma\rangle$

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f) \;\Leftrightarrow\; \mathcal{L}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)).$$

Problem:

How to compare two simple Petri automata?

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Problem:

How to compare two simple Petri automata?

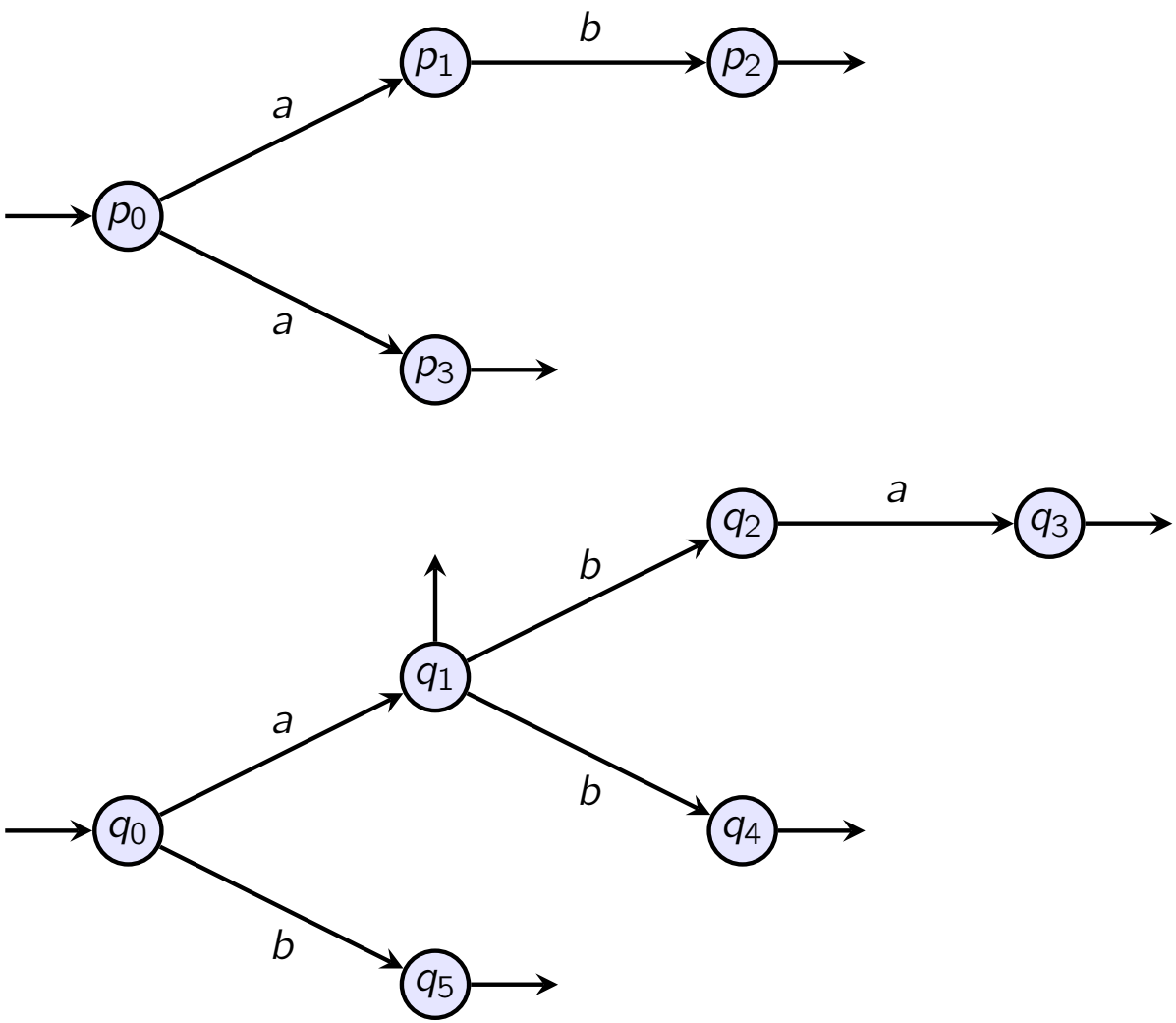
$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$ if and only if there is a simulation relation

$$\preccurlyeq \subseteq \mathcal{P}(P_1) \times \mathcal{P}(P_2 \rightharpoonup P_1)$$

between the configurations of $\mathcal{A}_1$ and the partial maps from the places of $\mathcal{A}_2$ to the places of $\mathcal{A}_1$.

Simulations - non-deterministic finite automata

$$\preccurlyeq \subseteq Q_1 \times \mathcal{P}(Q_2)$$



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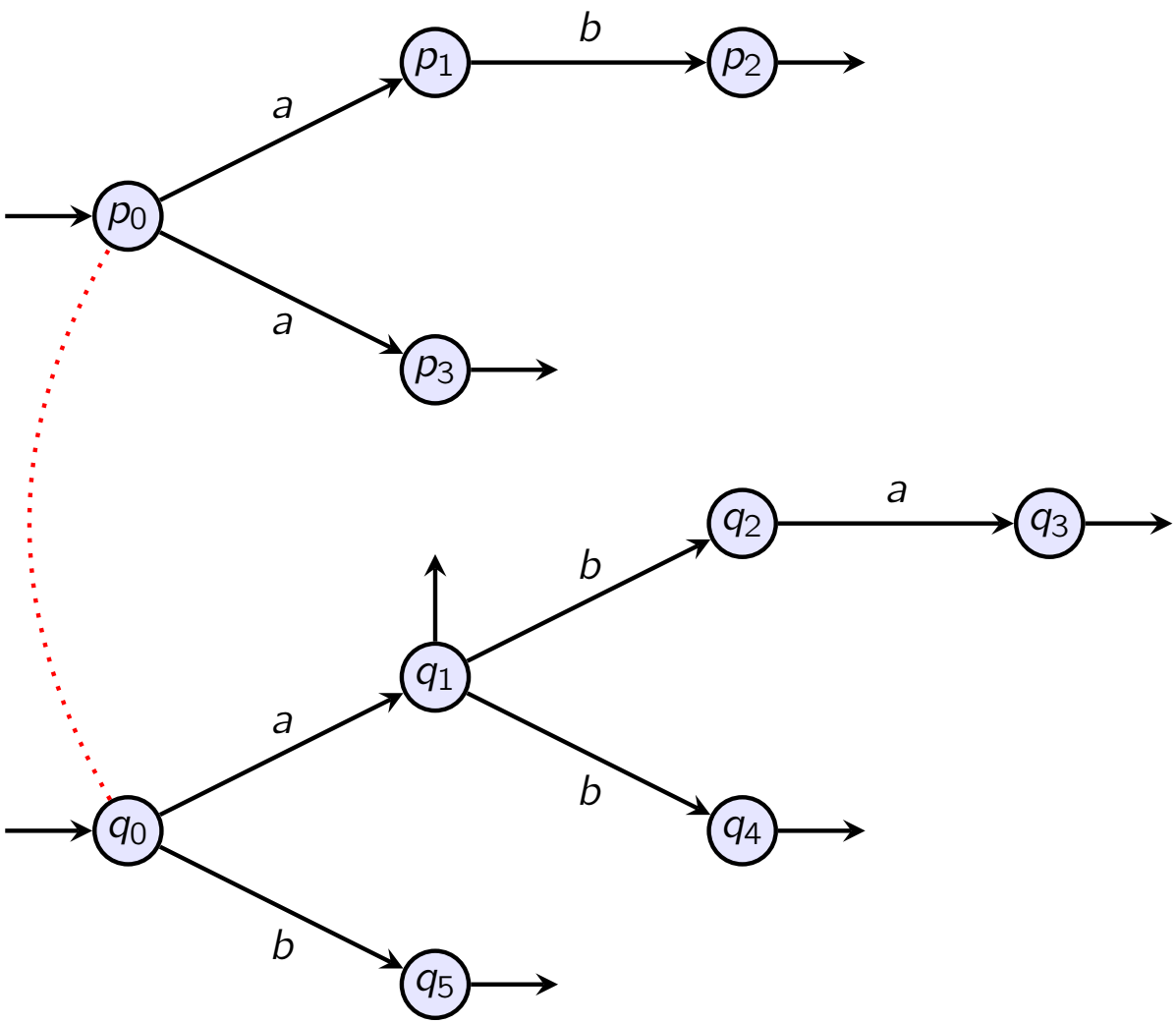

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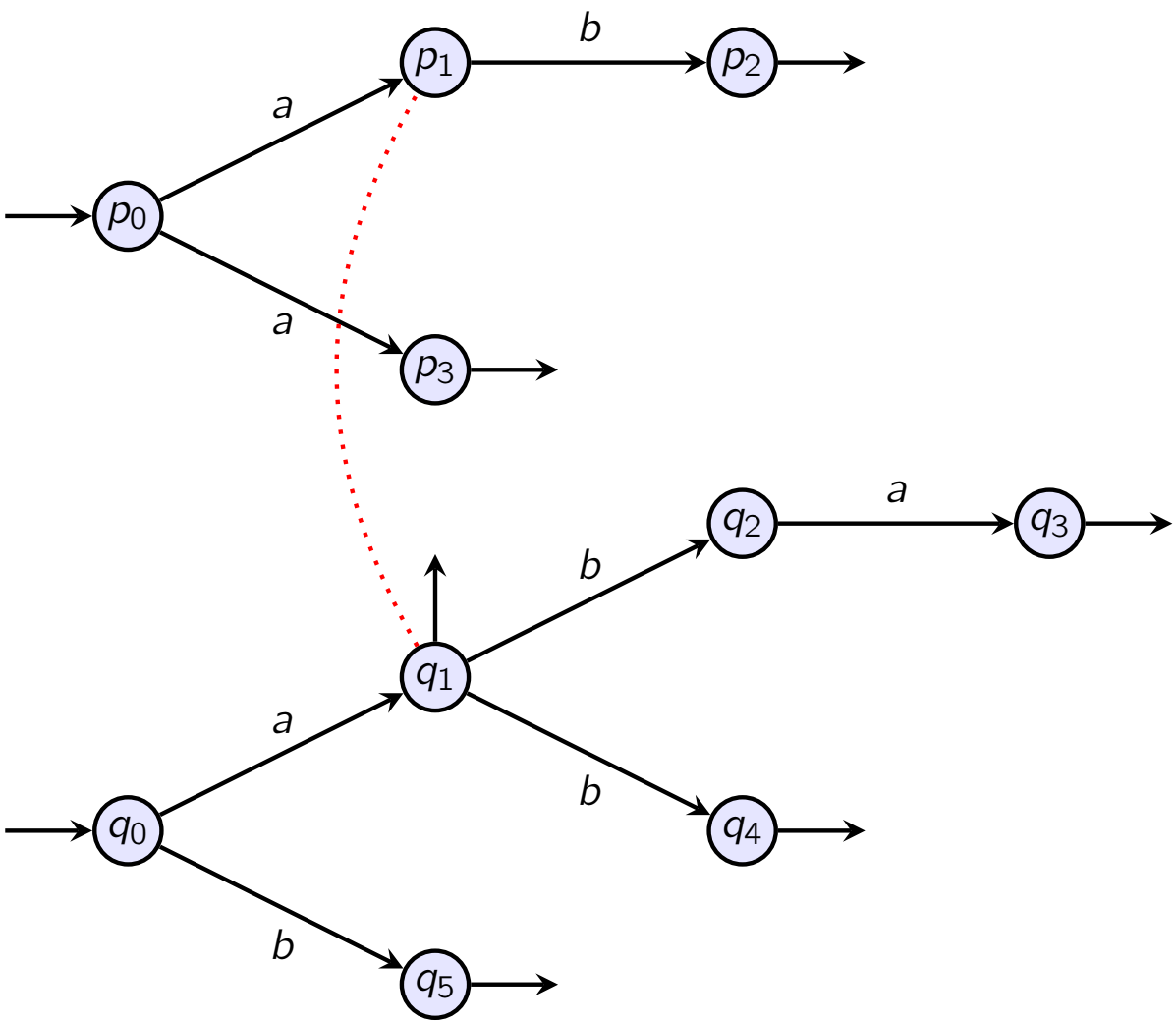

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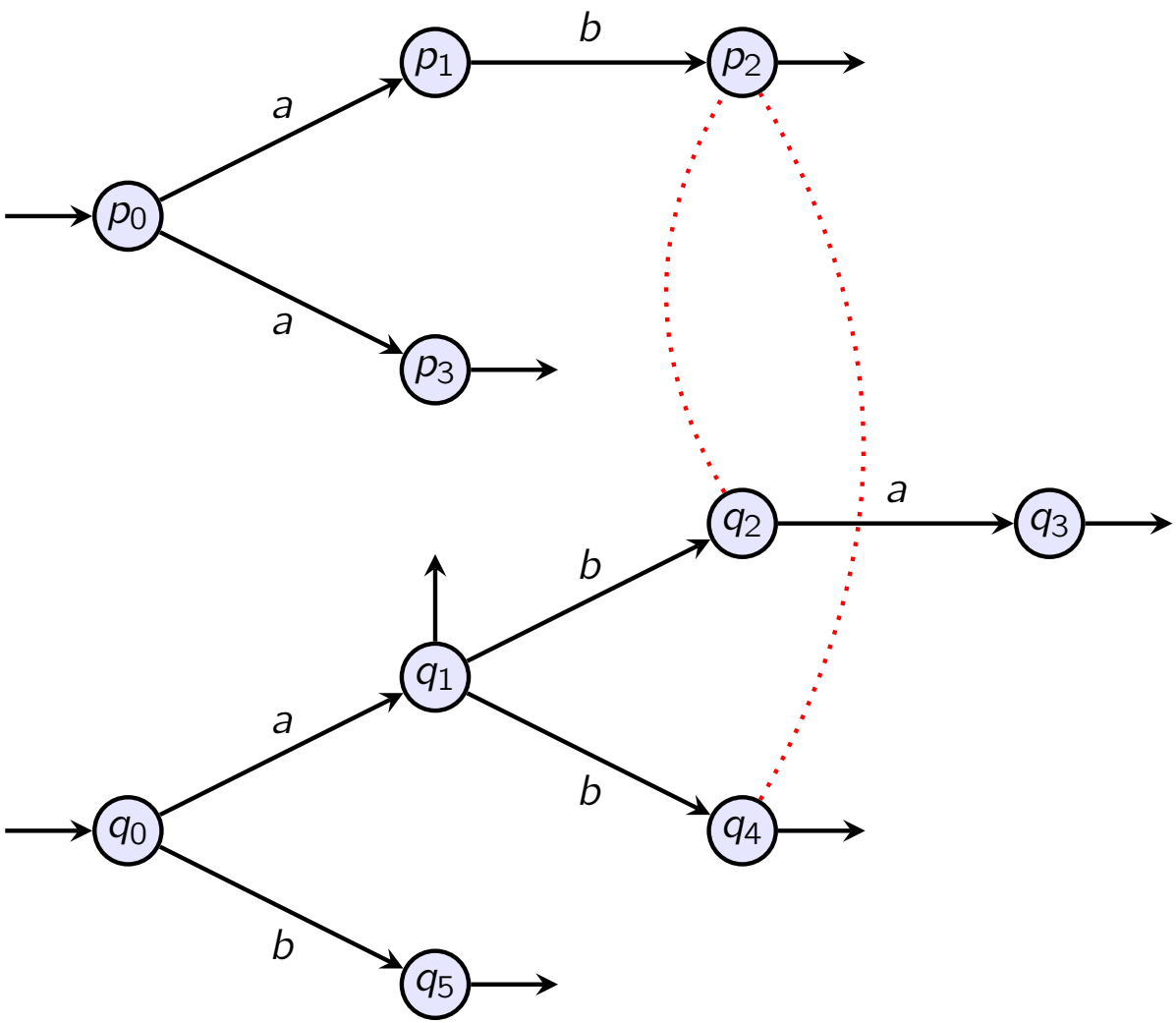

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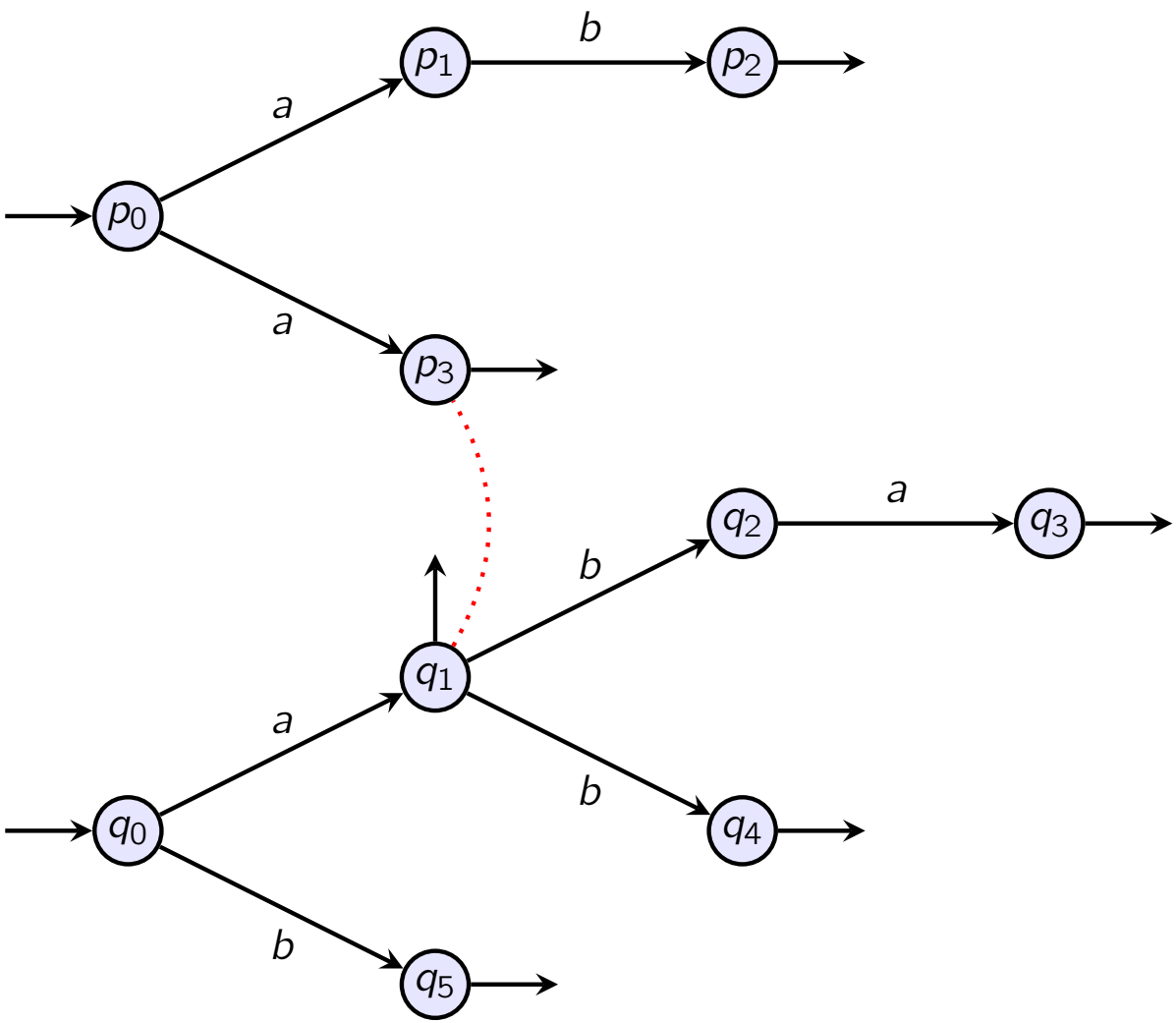

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Simulation - Petri automata

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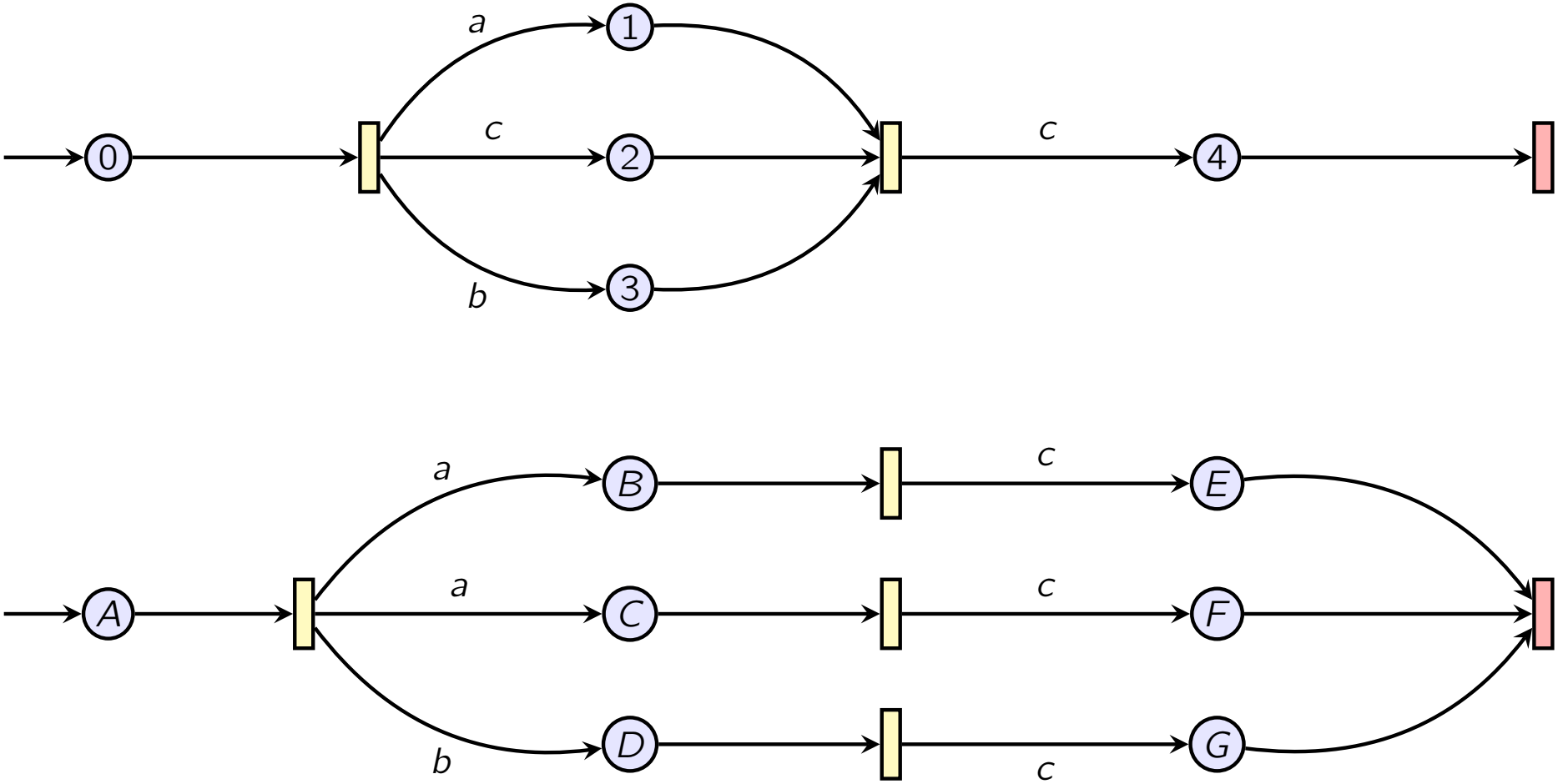

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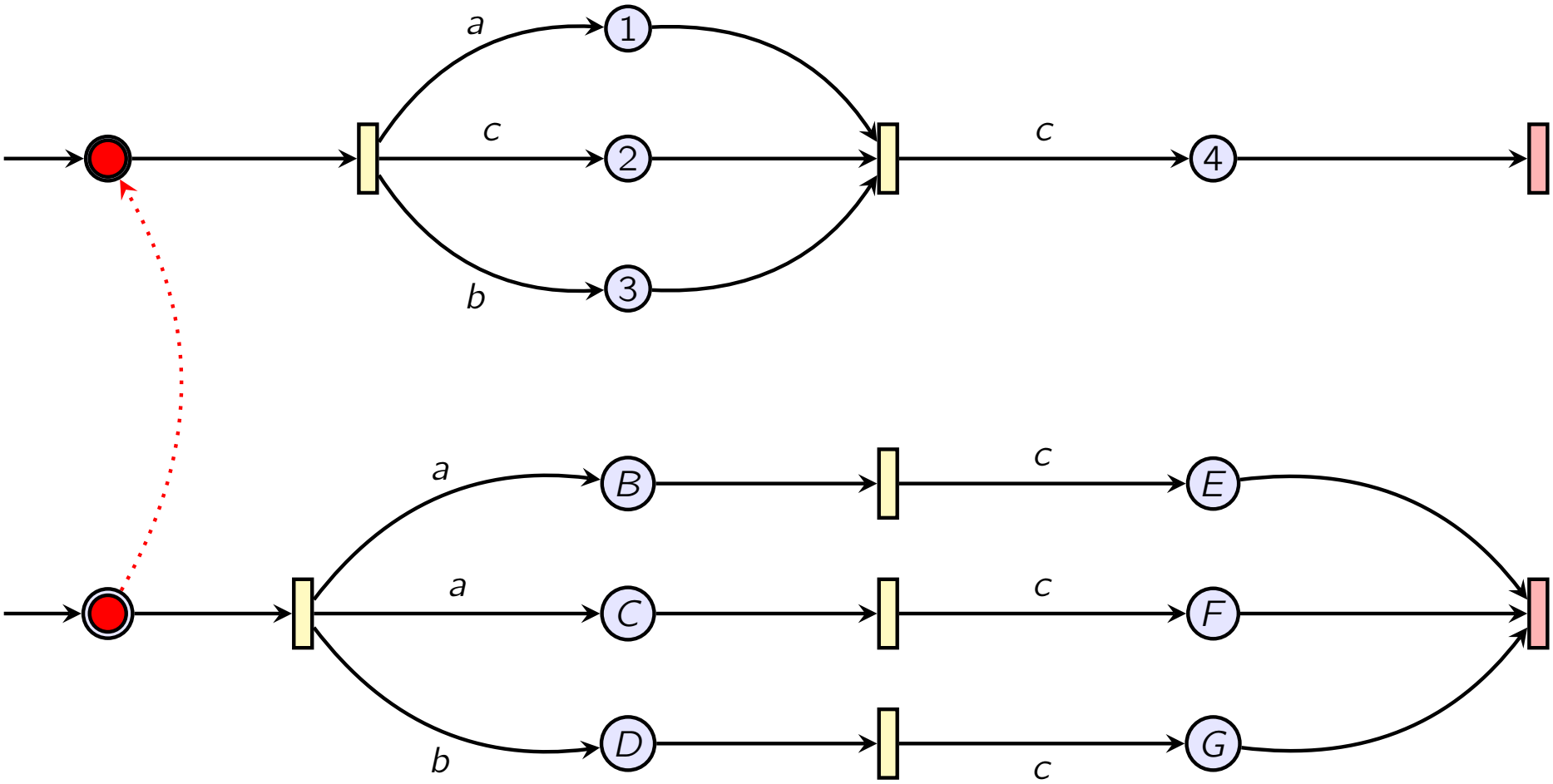

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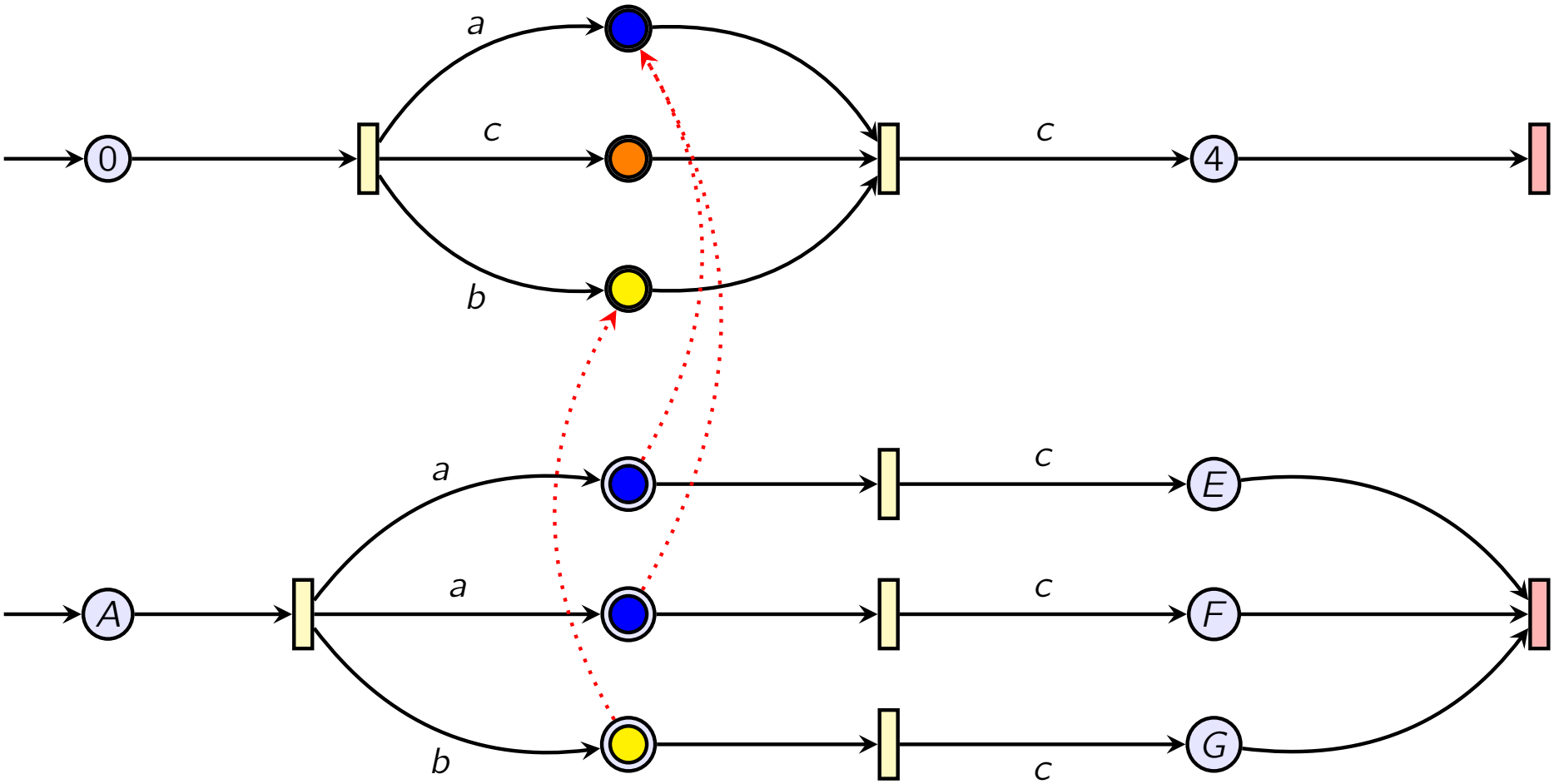

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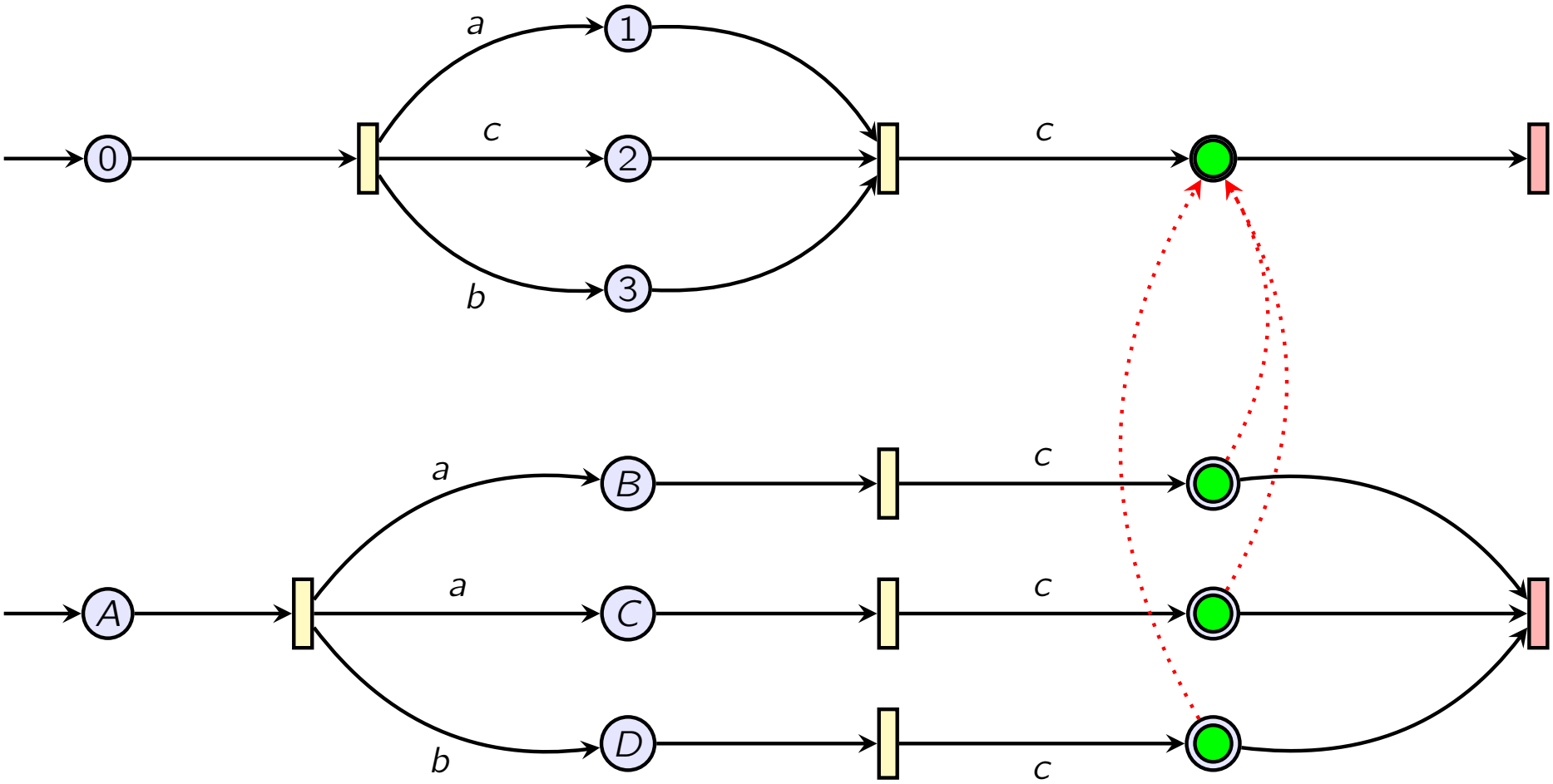

Comparing Petri automata

Theorem

Comparing simple Petri automata is **ExpSpace-complete**.

Proof.

ExpSpaceeasy: testing for the existence of a simulation can be done in exponential space;

ExpSpacehard: reduction from the universality problem for regular expressions with squaring.

□

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Meyer & Stockmeyer, **The equivalence problem for regular expressions with squaring requires exponential space**, *1972*

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Corollary

Relational equivalence for Identity-free Kleene lattices is **decidable** in ExpSpace.

Complexity lower bound

$$e, f \in \mathrm{GReg}\langle\Sigma\rangle ::= 0 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^+$$

Theorem

Given an expression $e \in \mathrm{GReg}\langle\Sigma\rangle$, testing if $\mathcal{L}(e) = \Sigma^+$ is **ExpSpace-complete**.

Proof. Simple adaptation of the proof for regular expressions with intersection by Fürer:

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Corollary

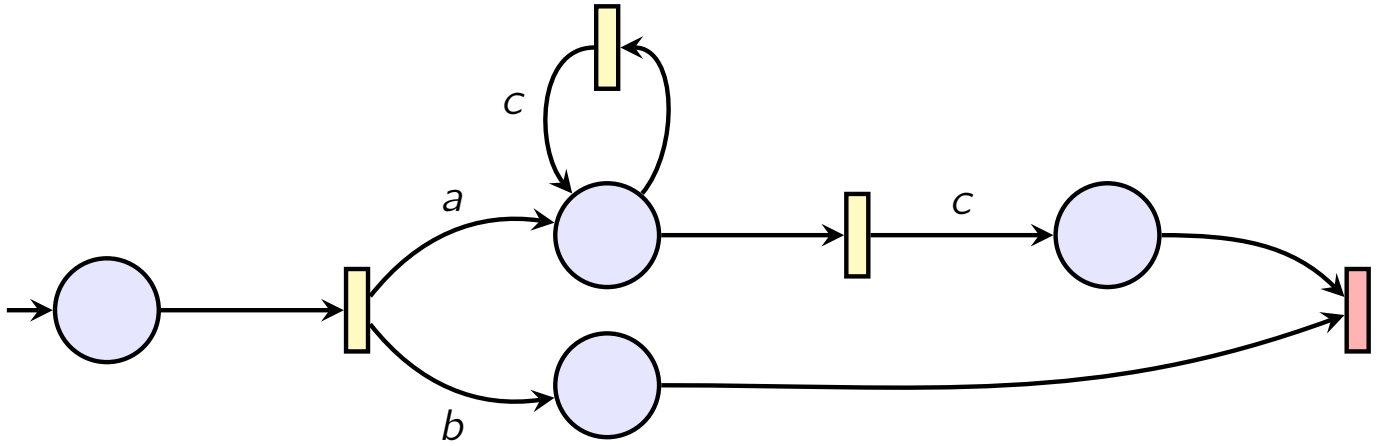
Relational equivalence for Identity-free Kleene lattices is **ExpSpace-hard**.

Proof. $\mathcal{L}(e) = \Sigma^+ \Leftrightarrow \mathcal{G}(\Sigma^+) \subseteq {}^{\triangleleft}\mathcal{G}(e)$. □

B. & Pous, **Petri automata**, *submitted in LMCS'17*

Outline

Trace language of an automaton



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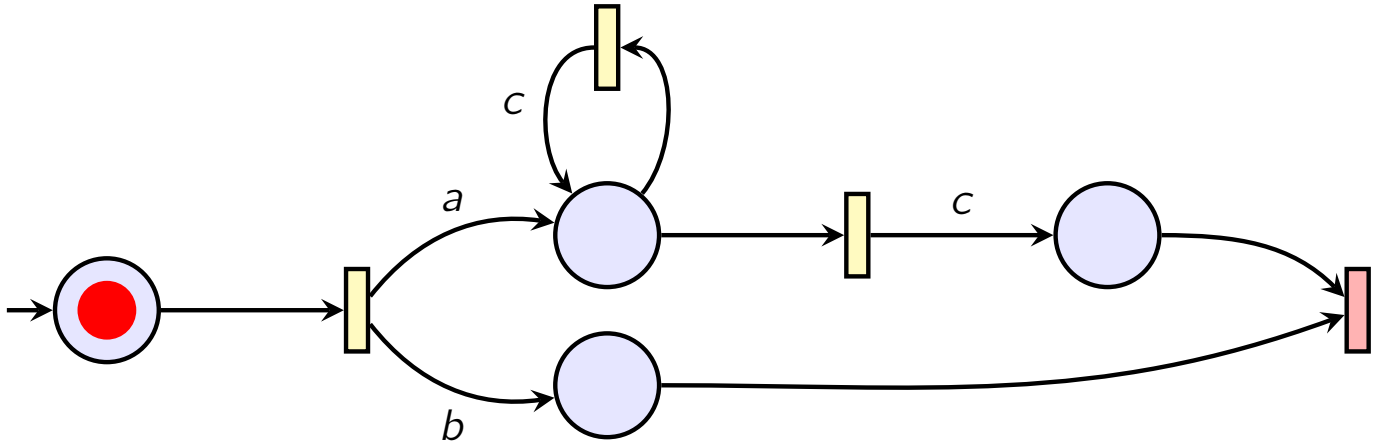

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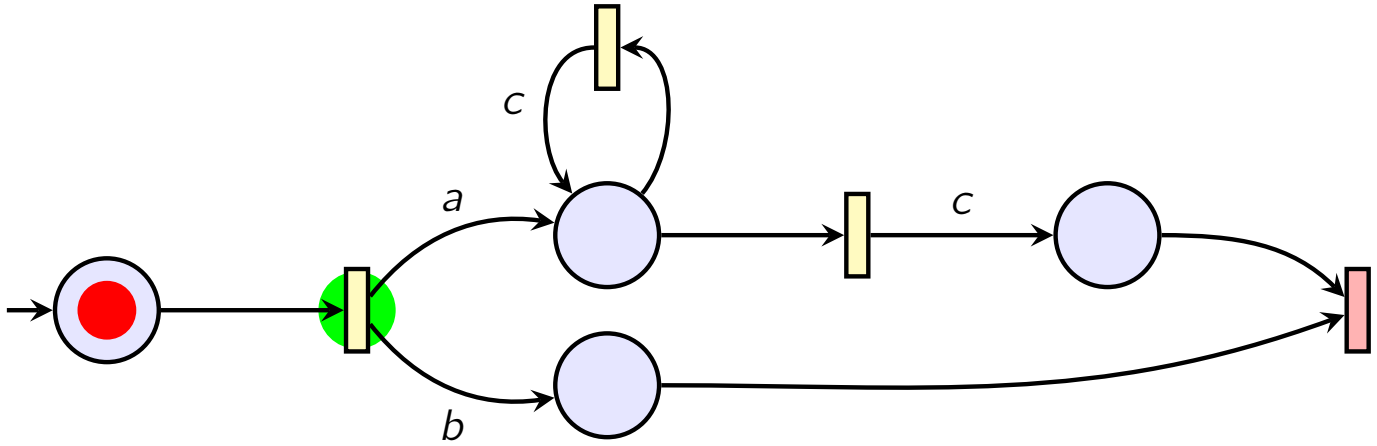

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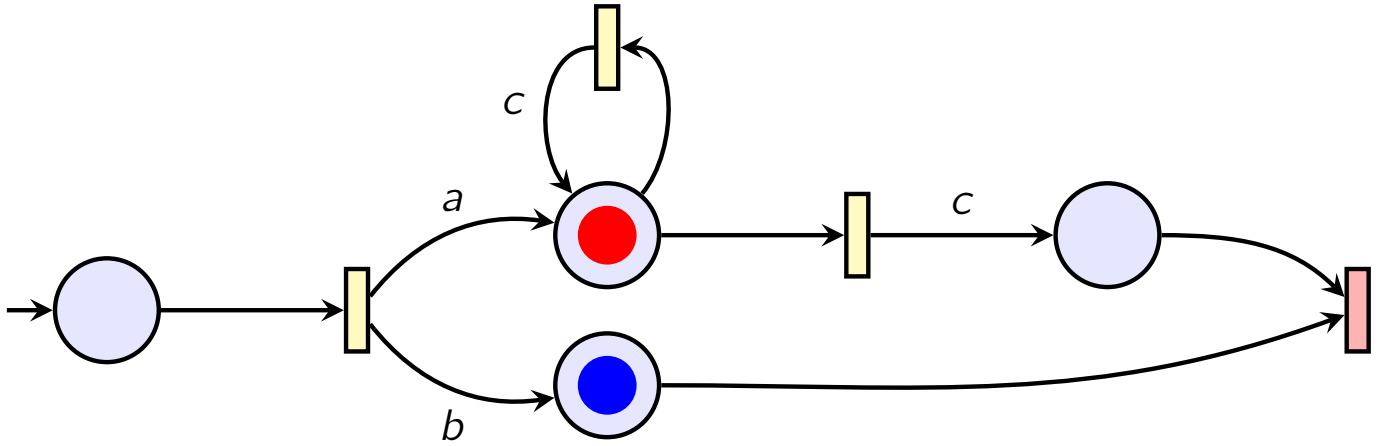

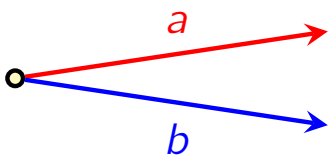

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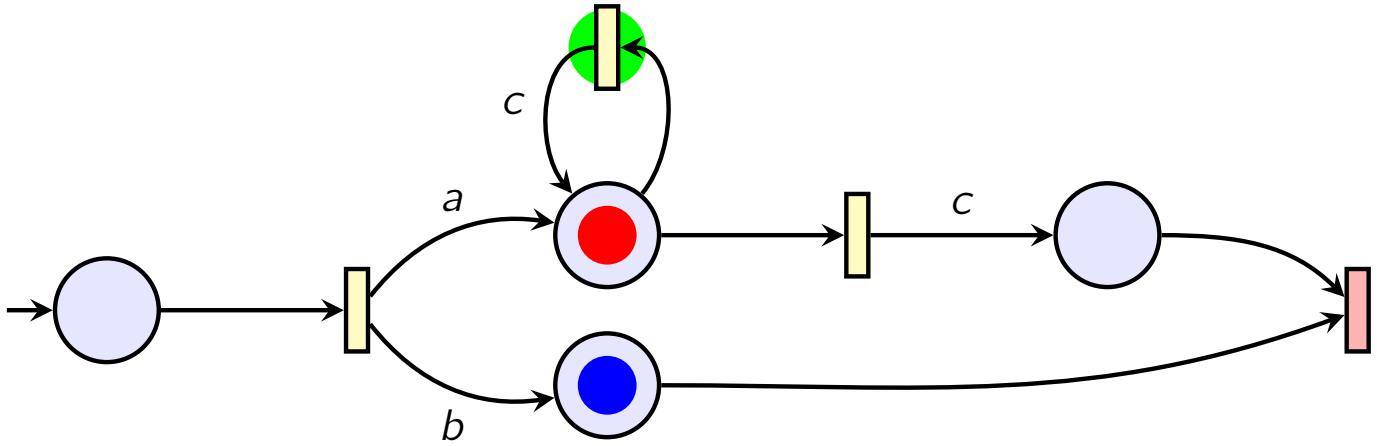

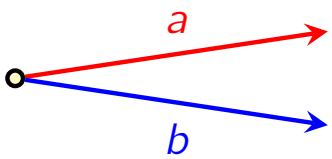

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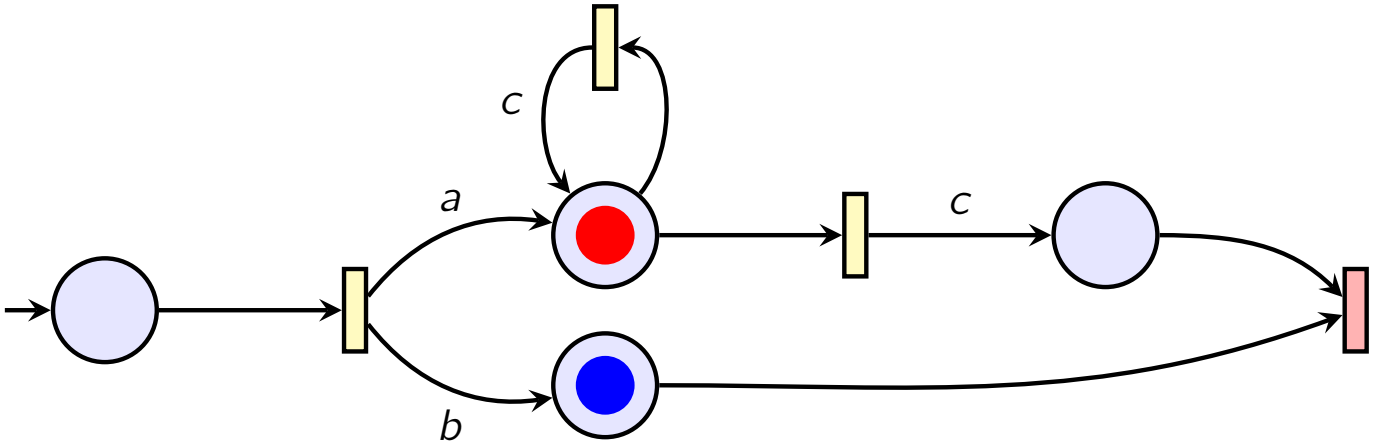

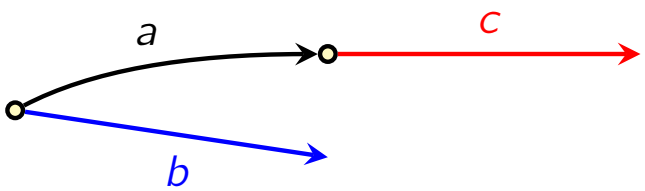

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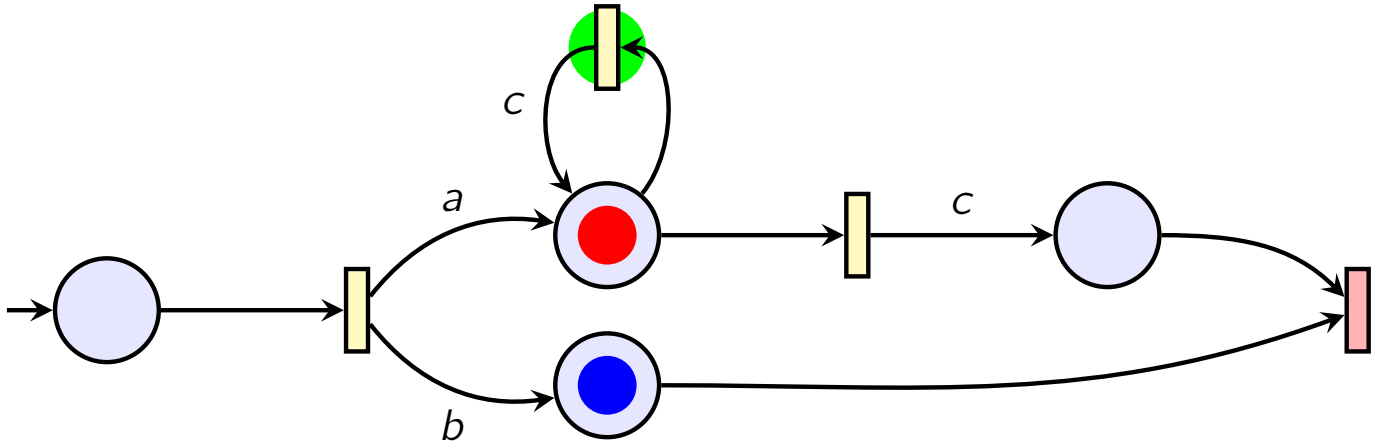

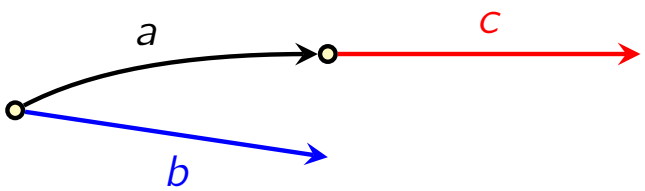

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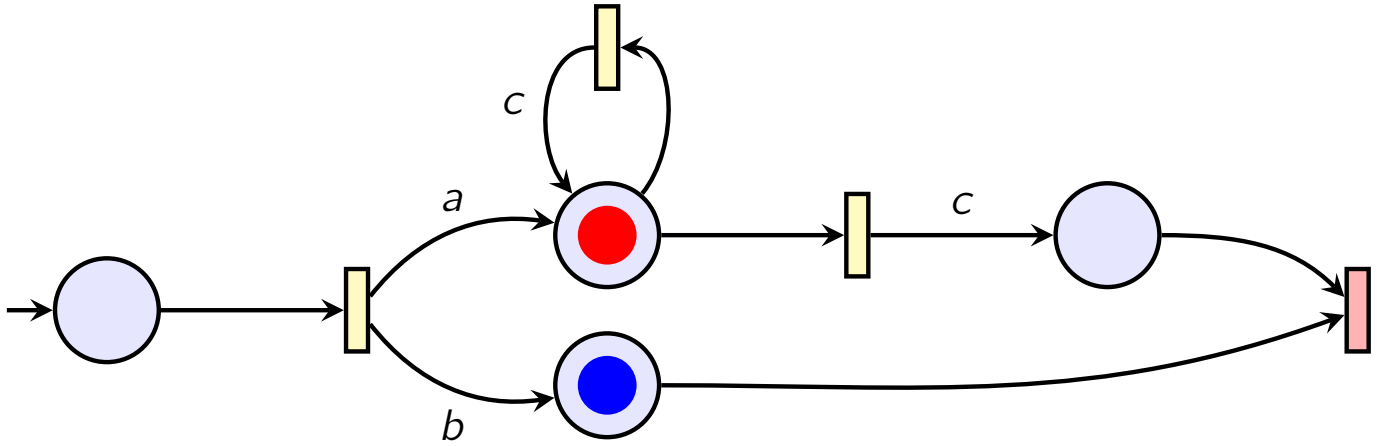

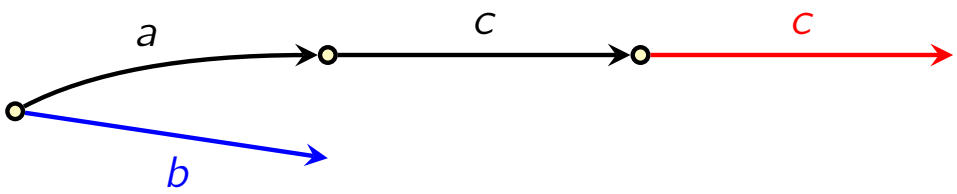

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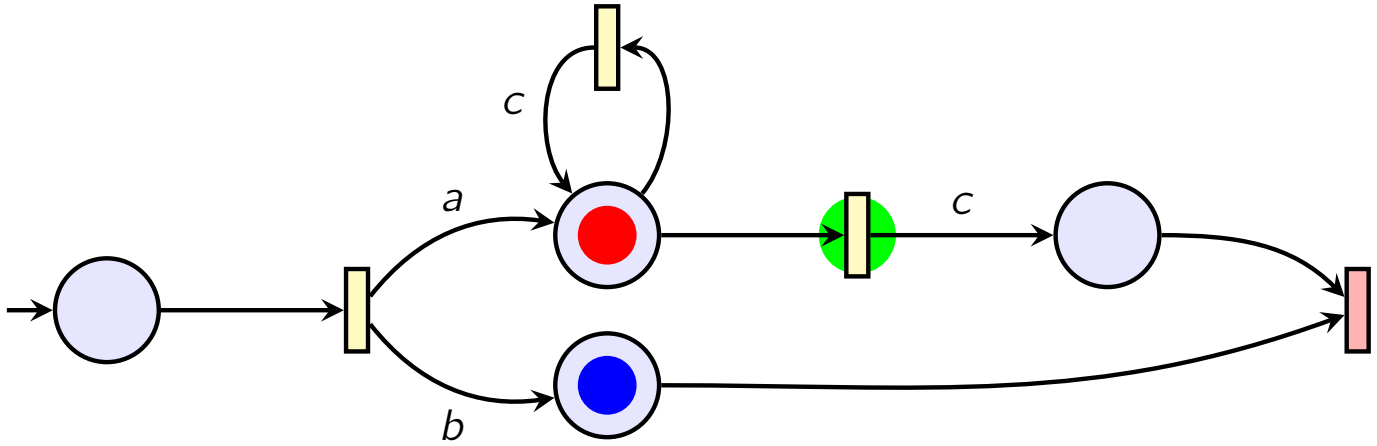

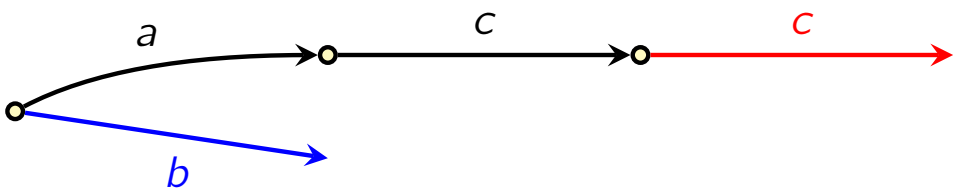

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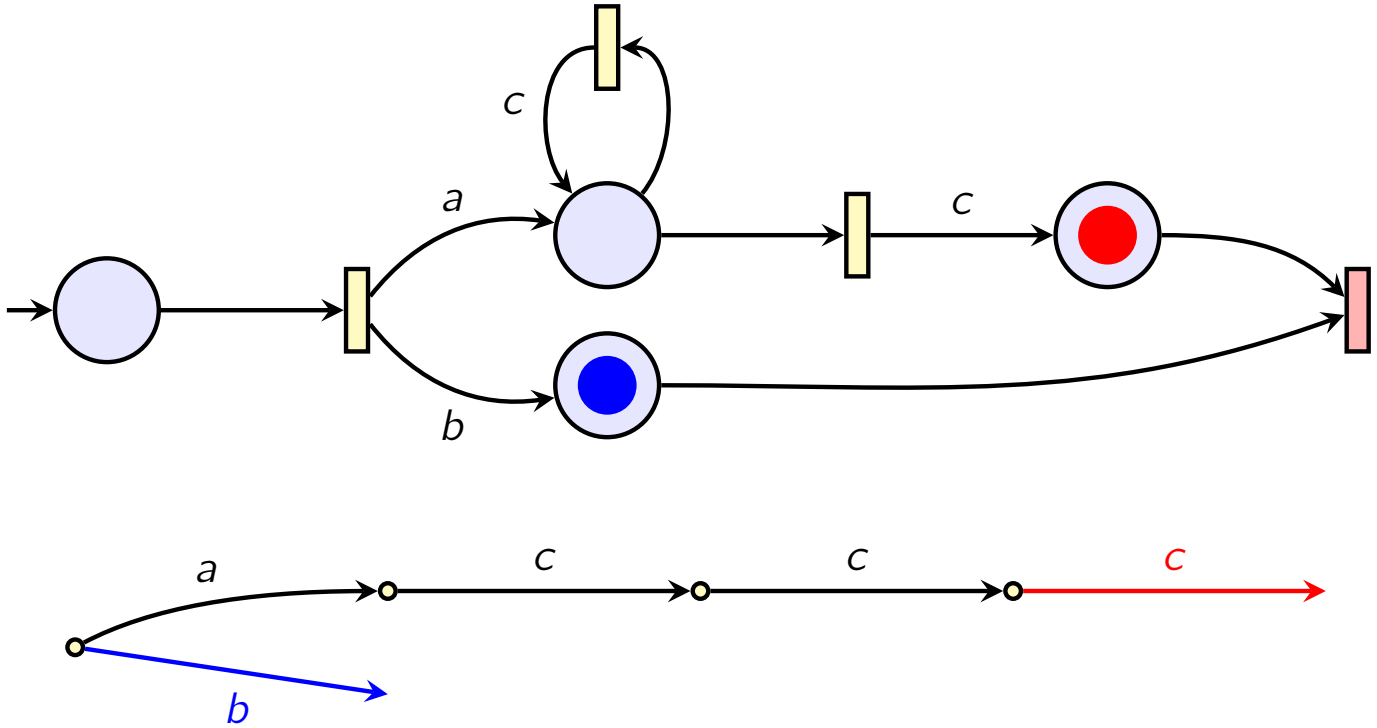

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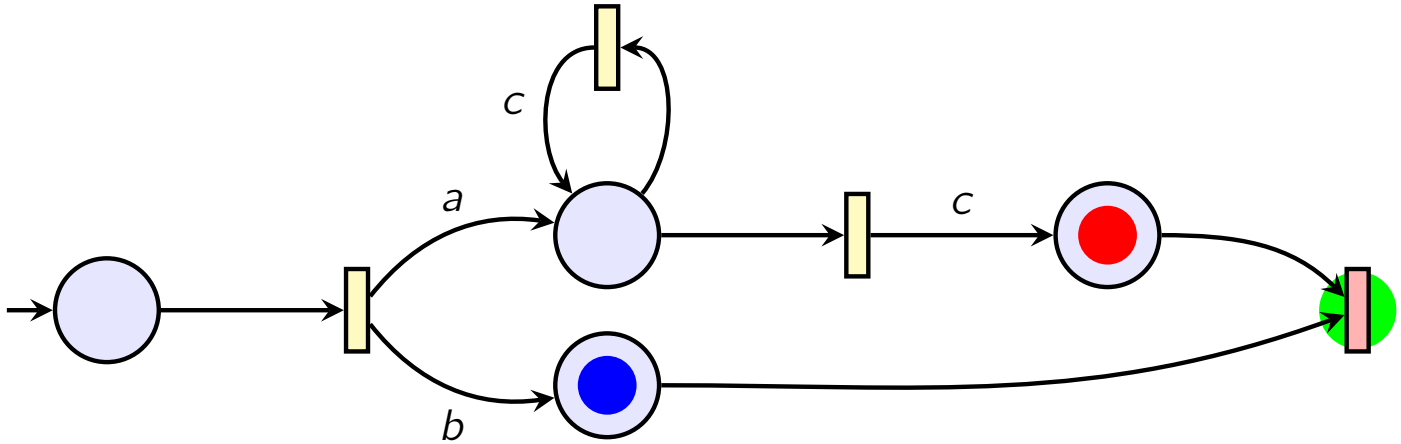

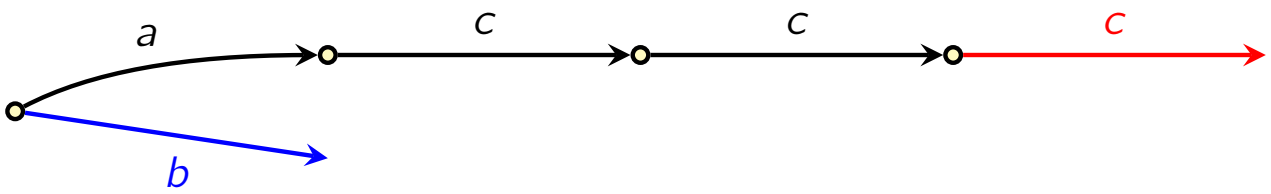

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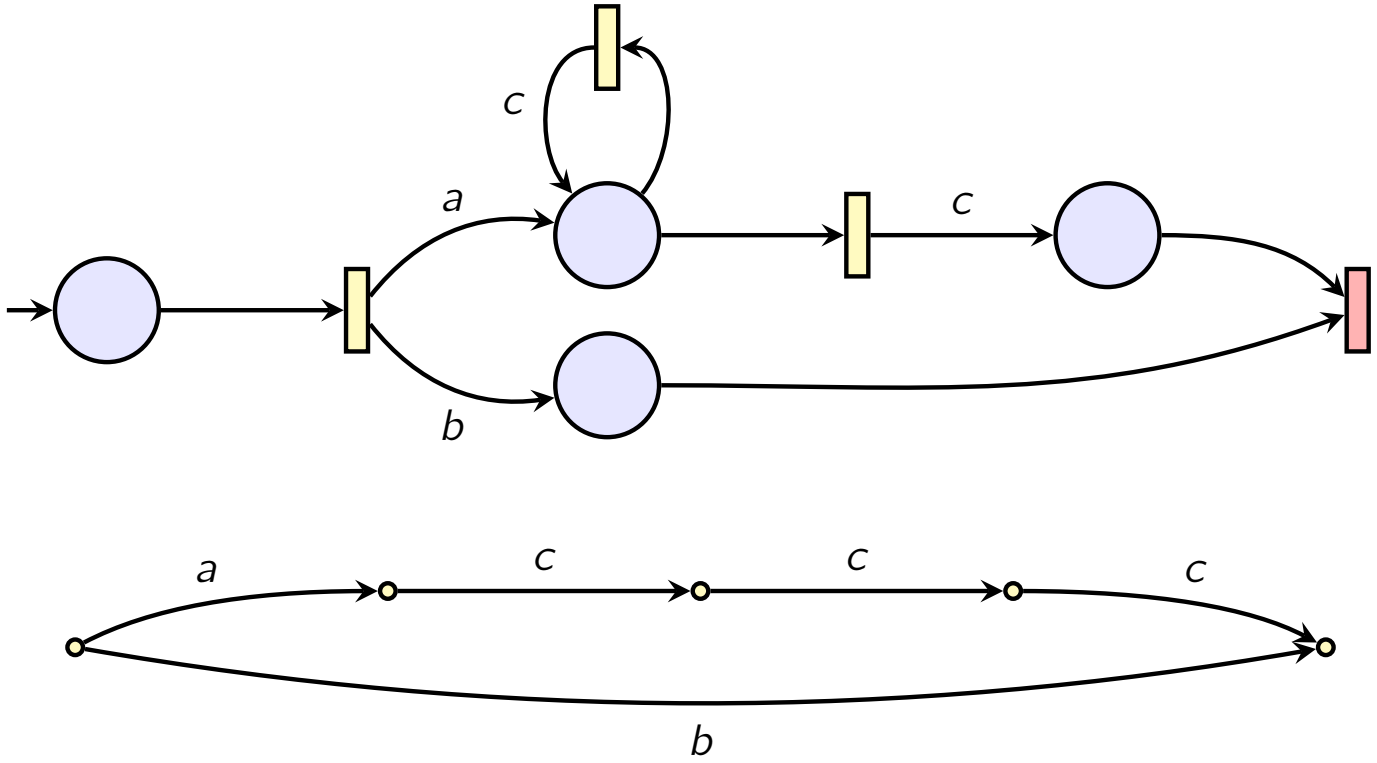

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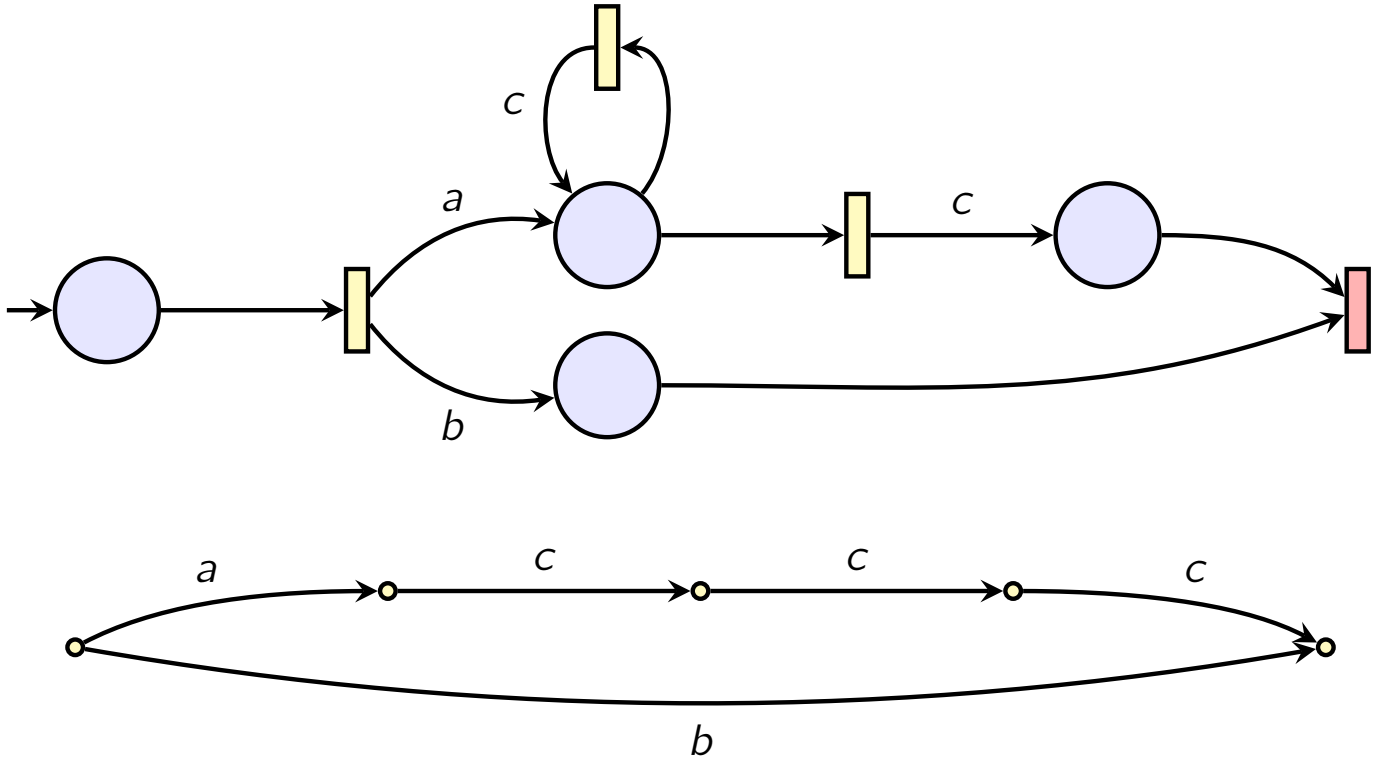
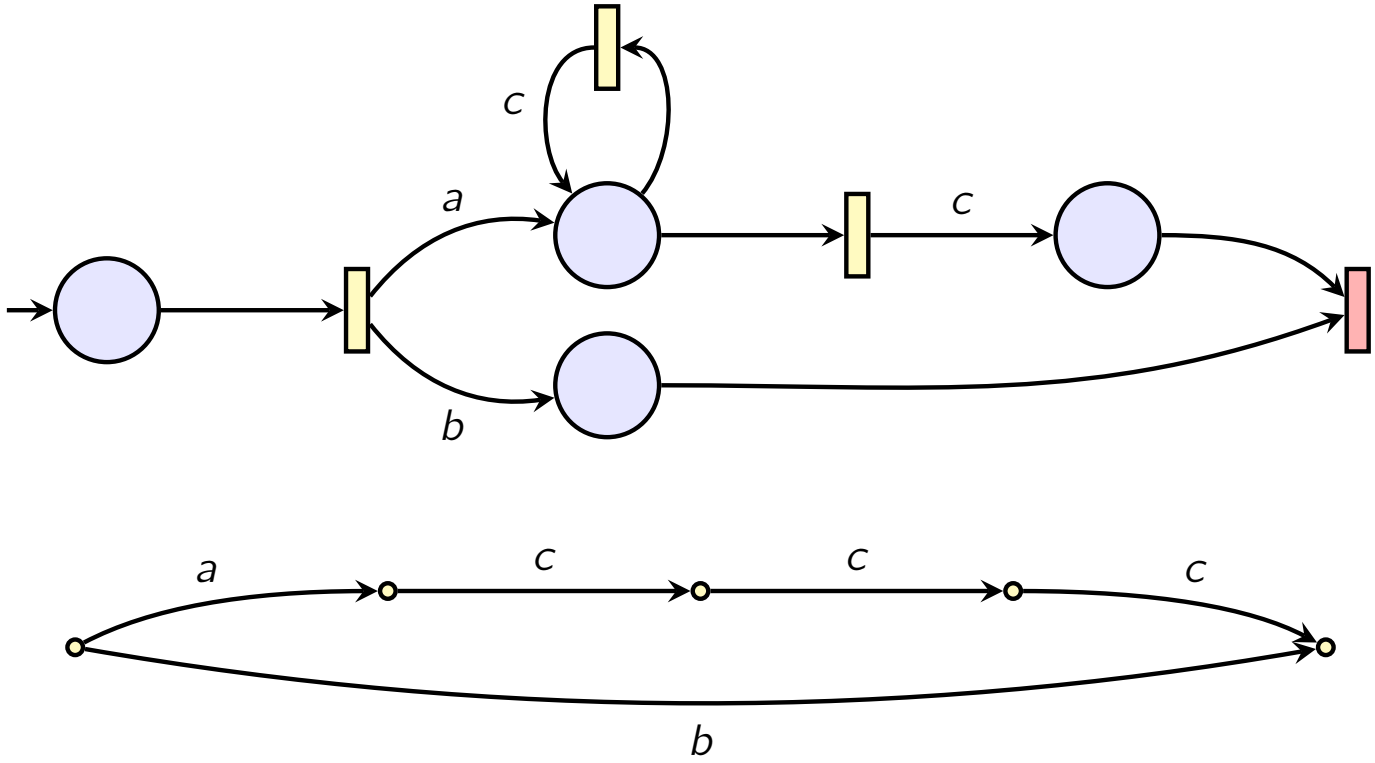


Trace language of $\mathcal{A}$

$$\mathcal{T}r(\mathcal{A}) = \{G \mid G \text{ is the trace of an accepting run in } \mathcal{A}\}$$

Trace language of an automaton



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Here: $\mathcal{T}r(\mathcal{A}) = \mathcal{G}((a \cdot c^+) \cap b)$.

Traces, languages and expressions

Lemma

If $\mathcal{A}$ is only labelled with Σ,

$$\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\mathcal{T}r(\mathcal{A}).$$

Fact

$e \in \mathrm{GReg}\langle\Sigma\rangle$,

$$\mathcal{T}r(\mathcal{A}(e)) = \mathcal{G}(e).$$

Kleene Theorem

Definitions

A set of SP graphs $\mathcal{G}$ is:

regular if there is an expression e in $\mathrm{GReg}\langle\Sigma\rangle$ such that $\mathcal{G}(e) = \mathcal{G}$;

recognisable if there is a simple Petri automaton $\mathcal{A}$ such that $\mathcal{T}r(\mathcal{A}) = \mathcal{G}$.

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The class of regular sets of graphs coincides with the recognisable sets of graphs.

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Proof.

- **Expressions to automata**: inductive construction;

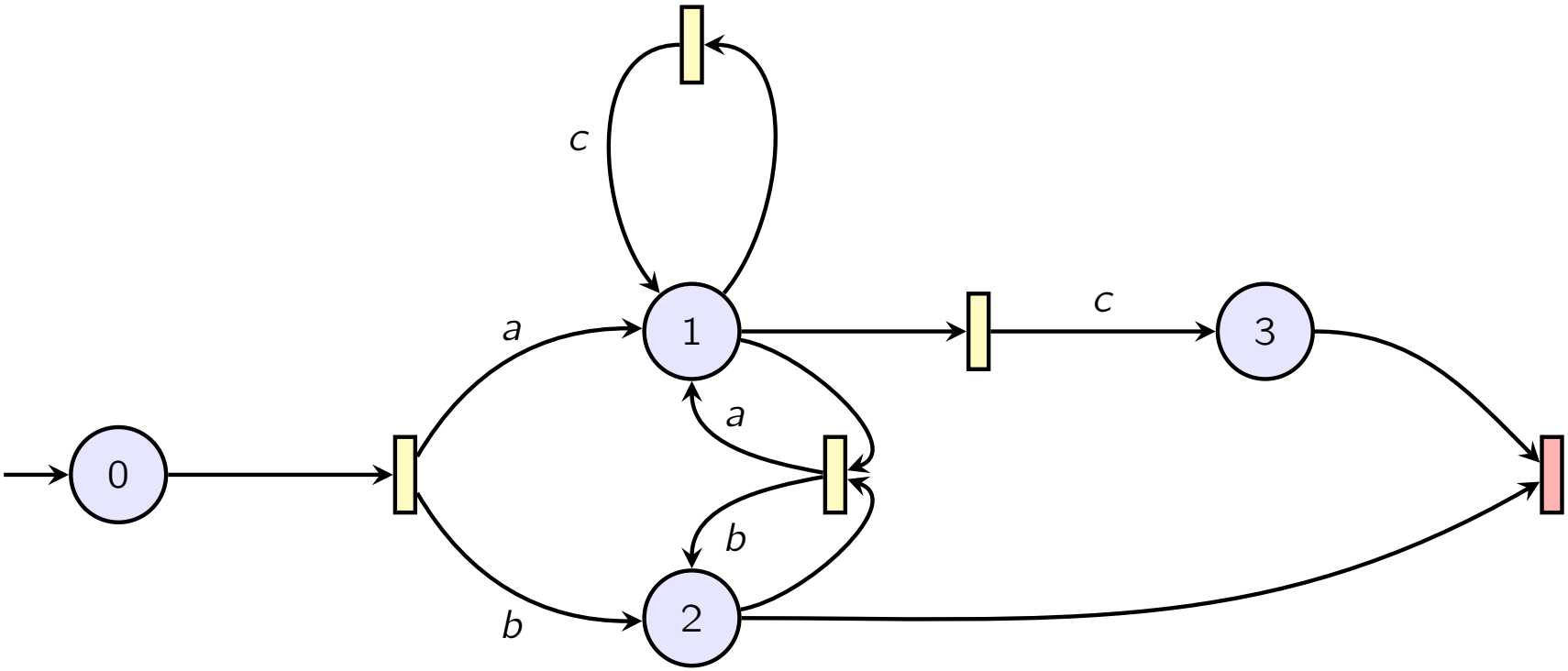
  B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

- **Automata to expressions**: next slide.

  B. & Pous, **Petri automata**, *submitted in LMCS'17*

□

From automata to expressions



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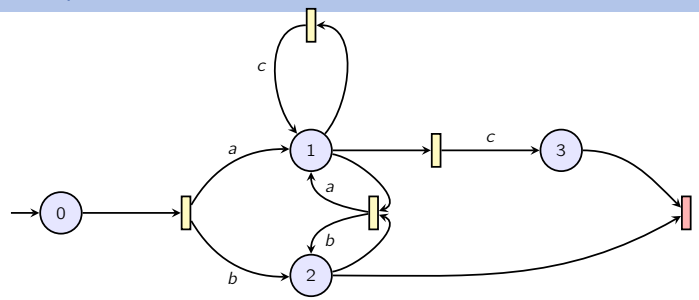

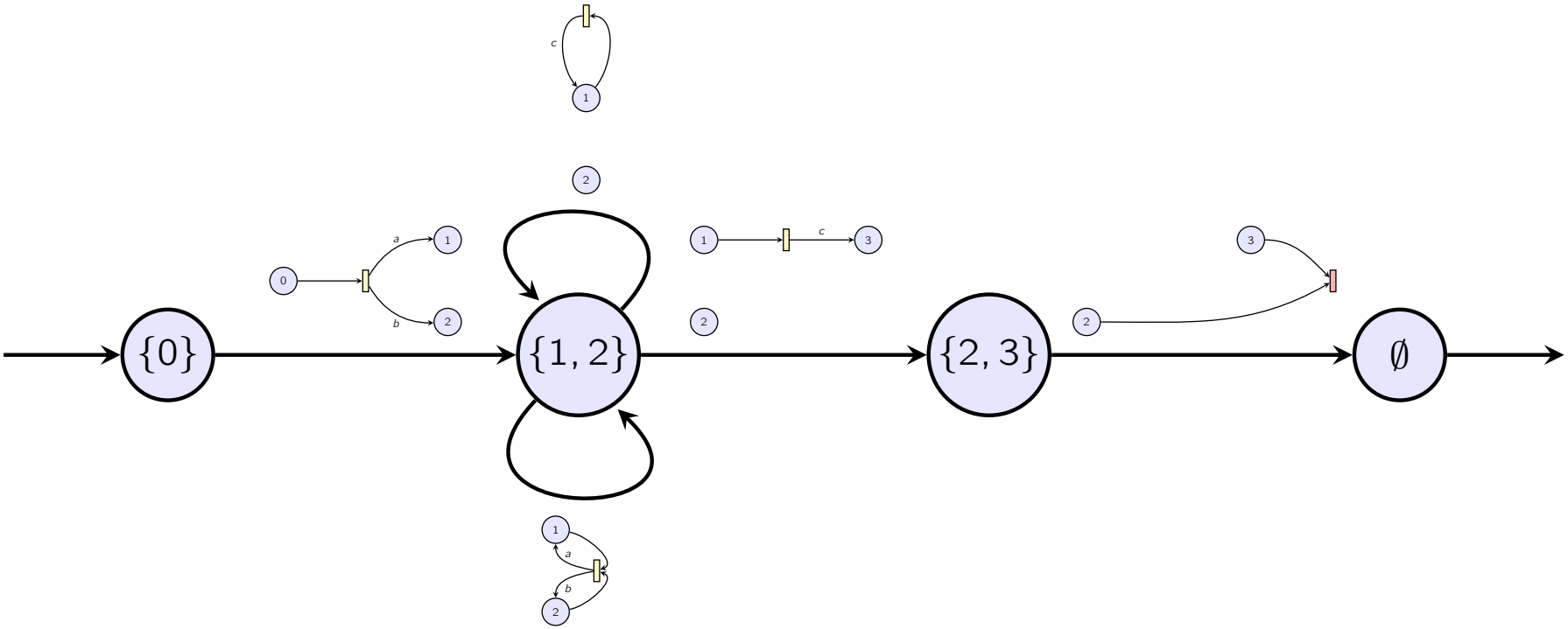

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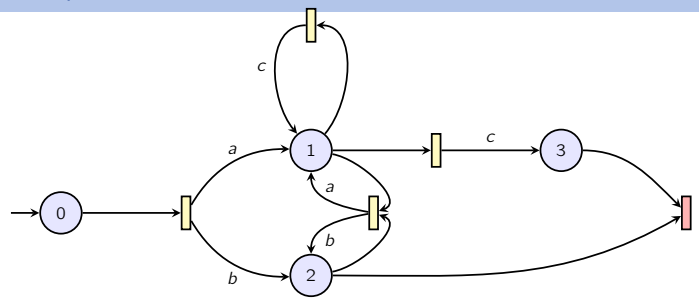
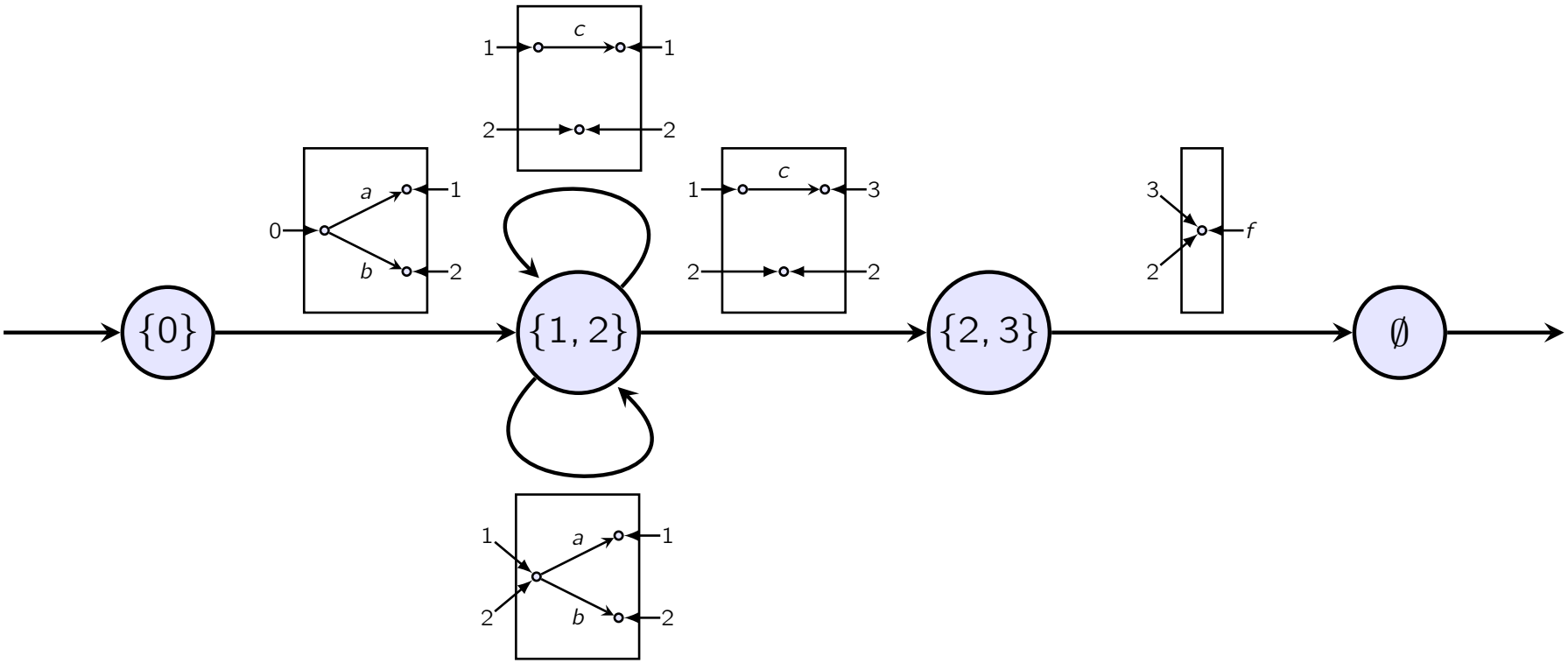
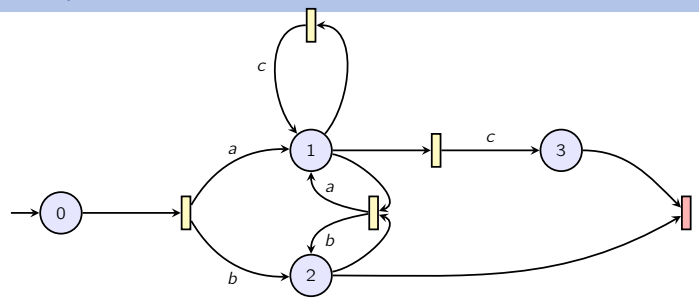
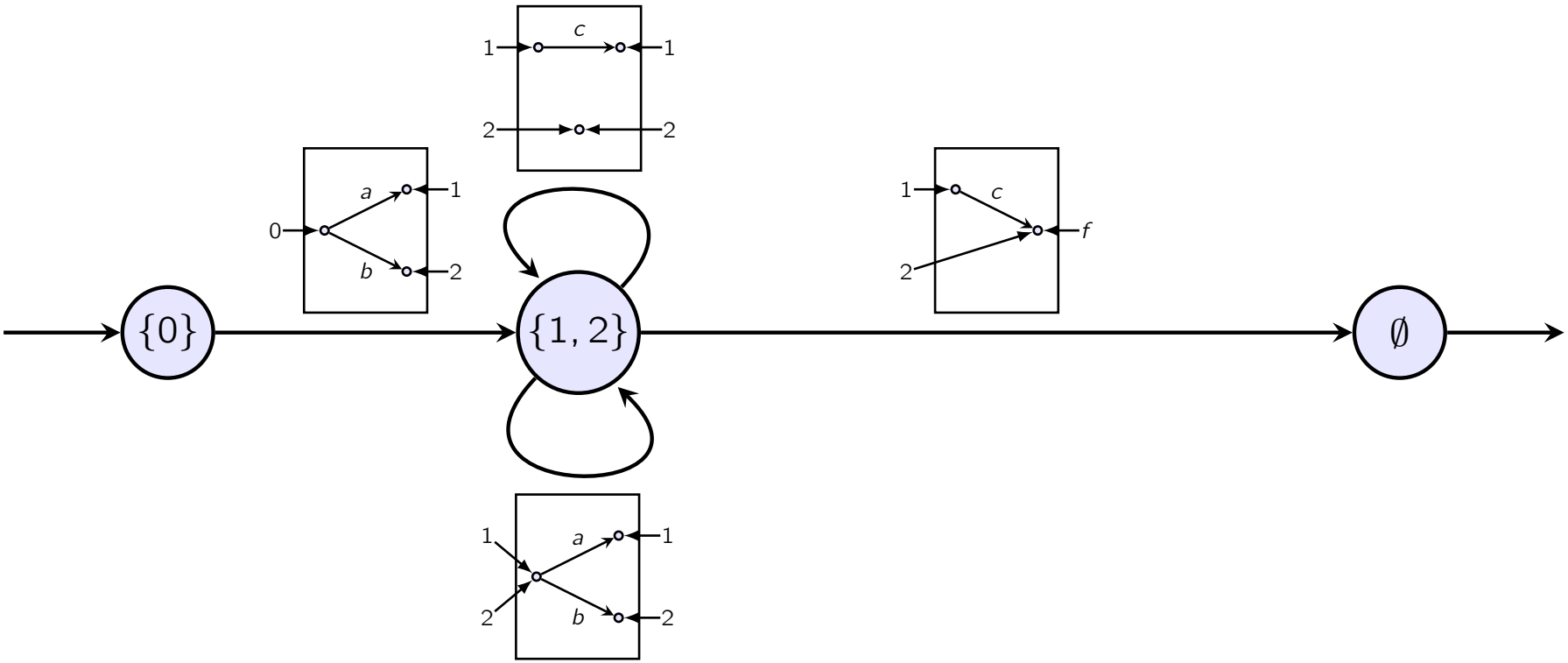




From automata to expressions





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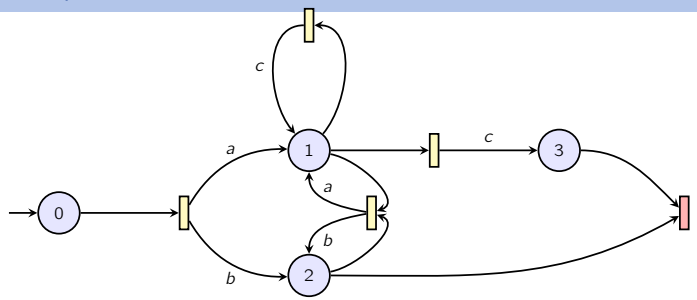

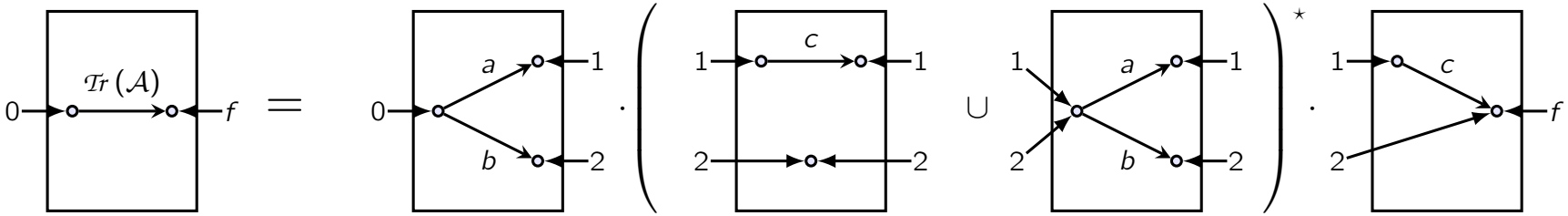

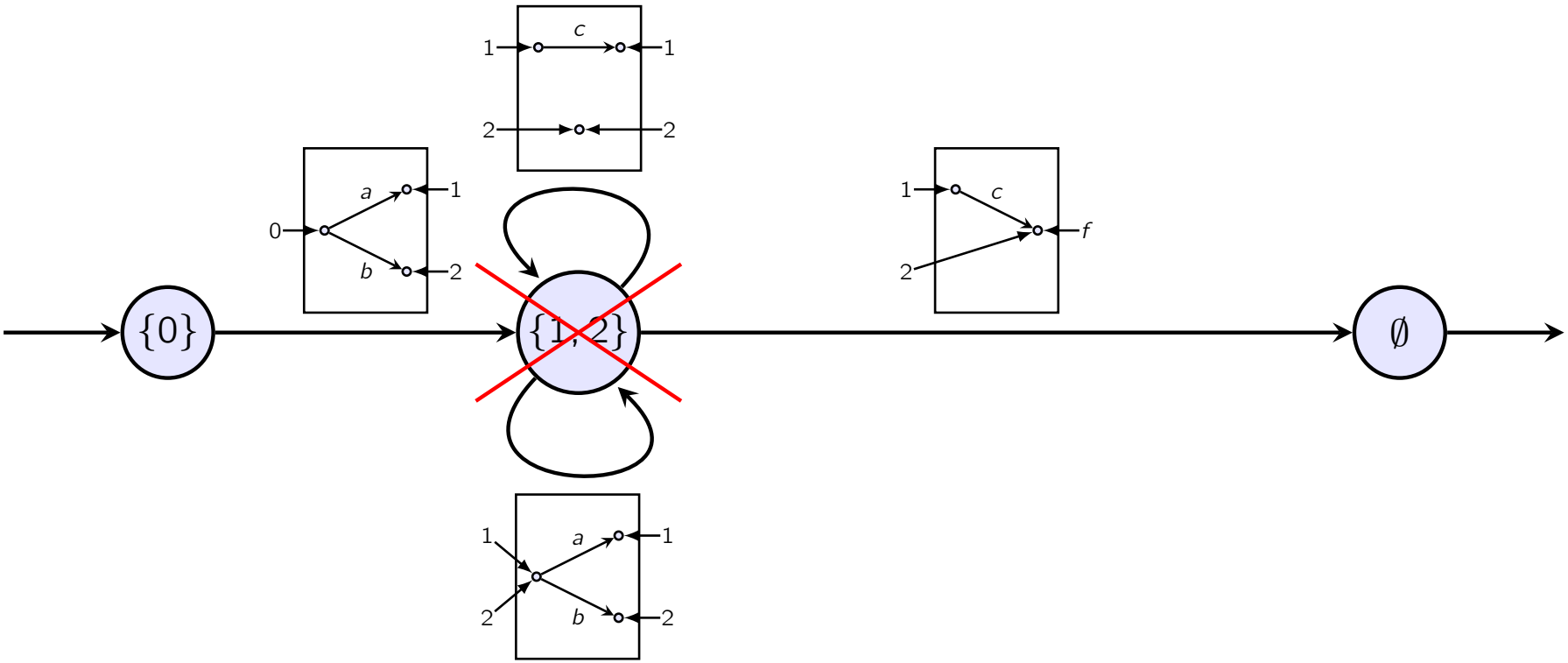

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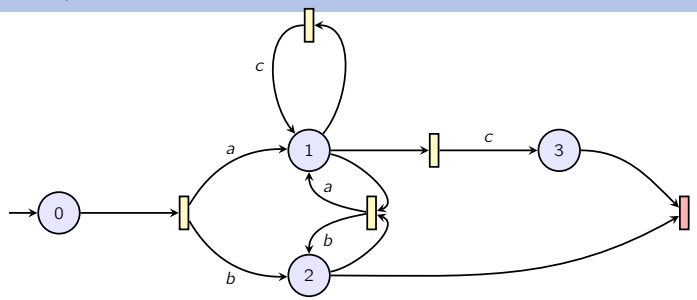
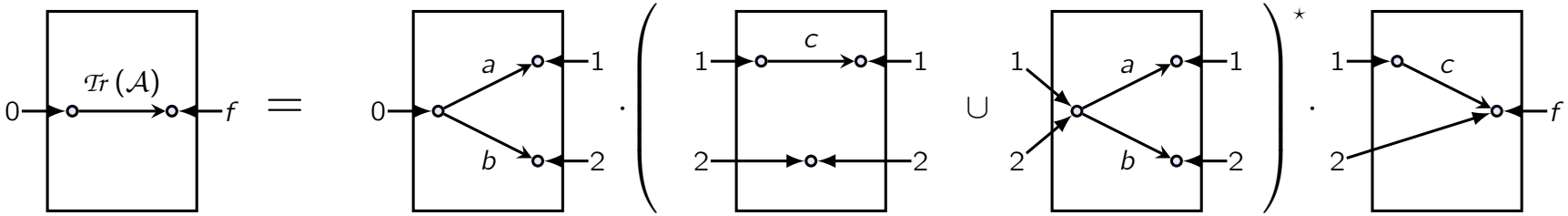
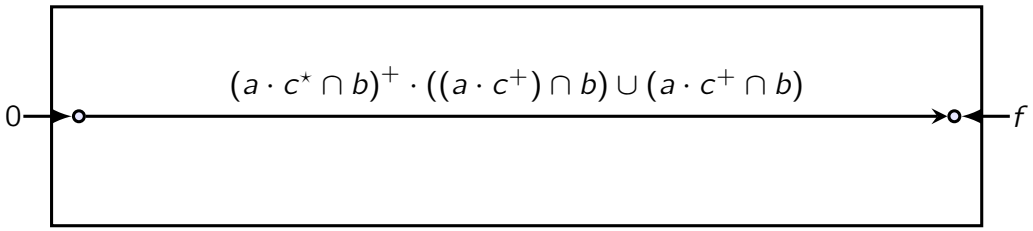


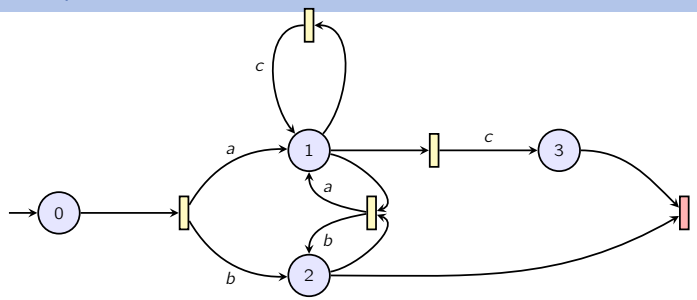


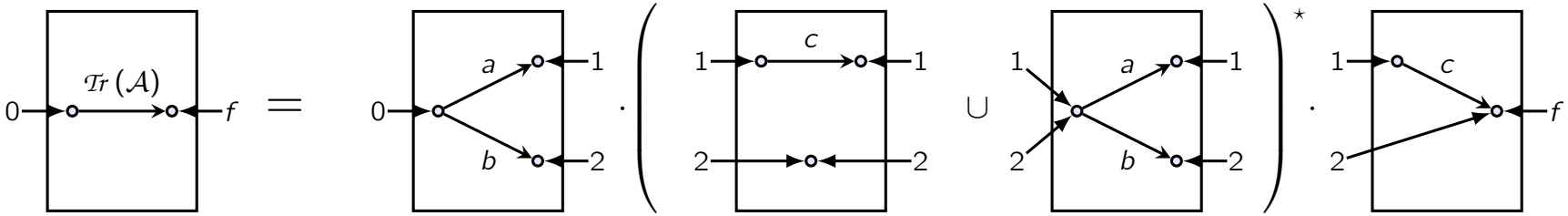
$\vdots$

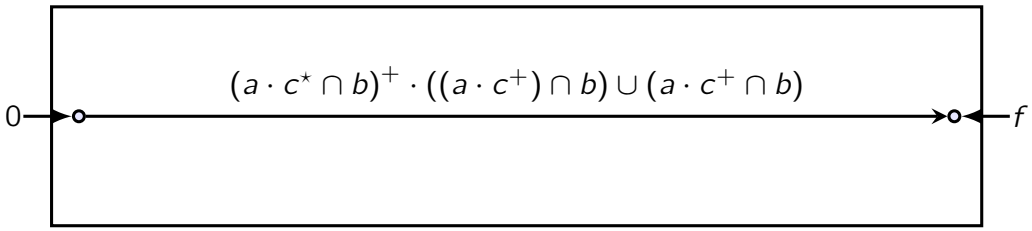


(With some effort...)

From automata to expressions







Result

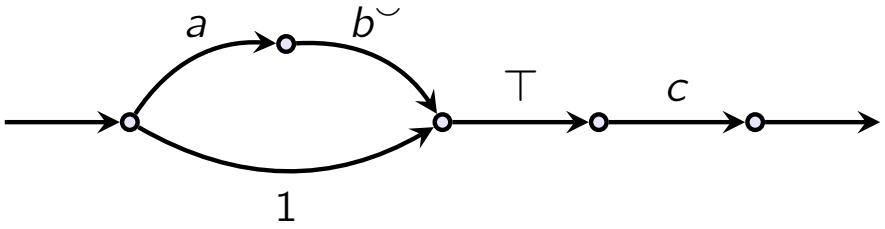
$$\mathcal{E}(\mathcal{A}) = (a \cdot c^\star \cap b)^+ \cdot \left(\left(a \cdot c^+\right) \cap b\right) \cup \left(a \cdot c^+ \cap b\right)$$

$$\mathcal{T}r(\mathcal{A}) = \mathcal{G}(\mathcal{E}(\mathcal{A}))$$

Outline

Relationship between $\mathcal{T}r(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$

Φ : graphs over Σ' $\rightarrow$ graphs over Σ.



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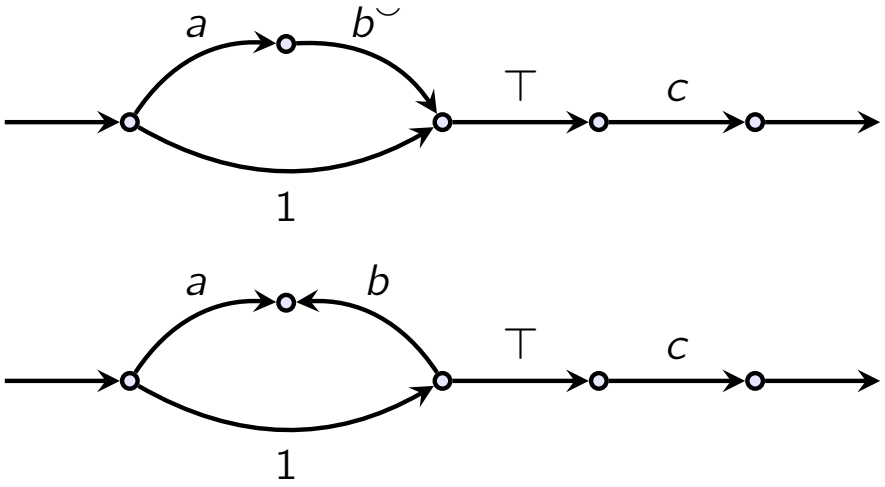

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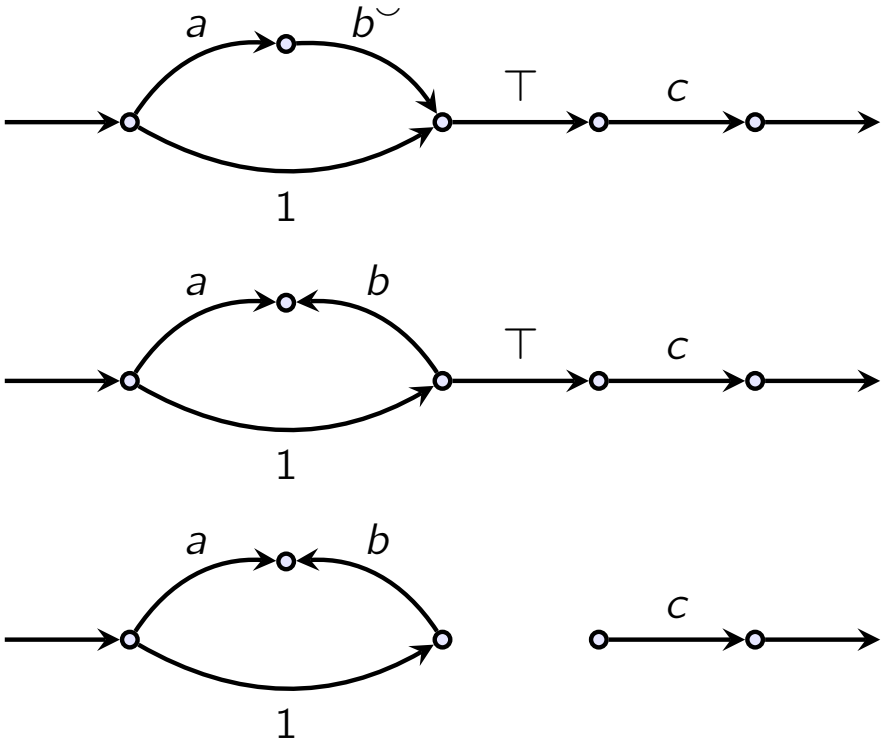

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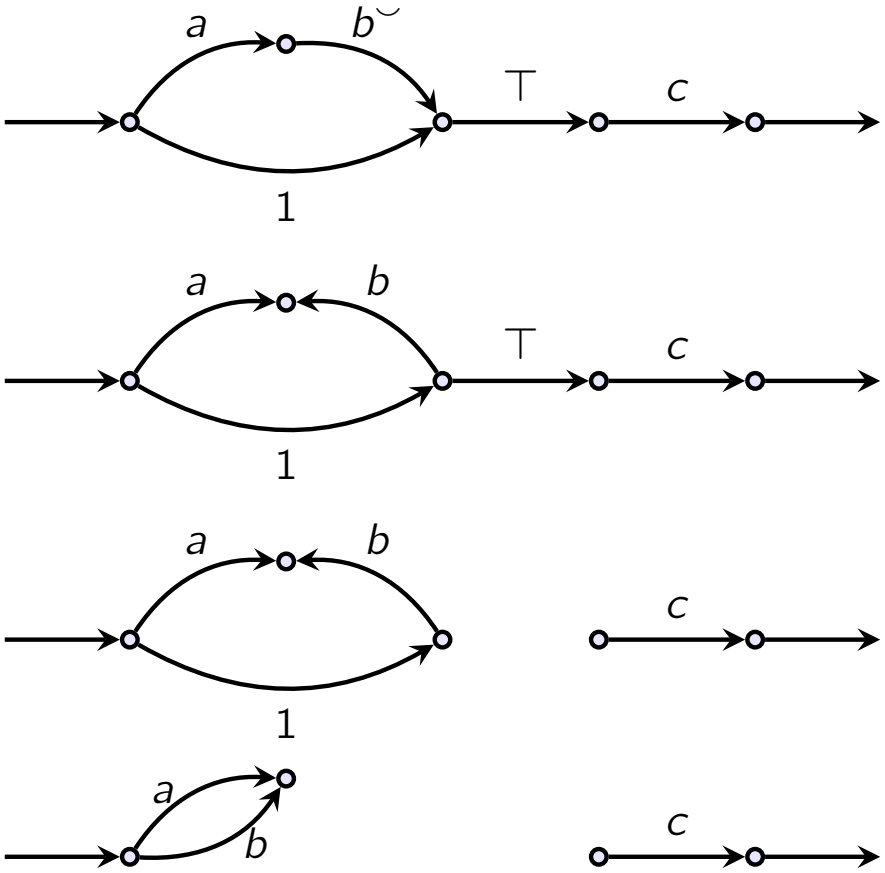

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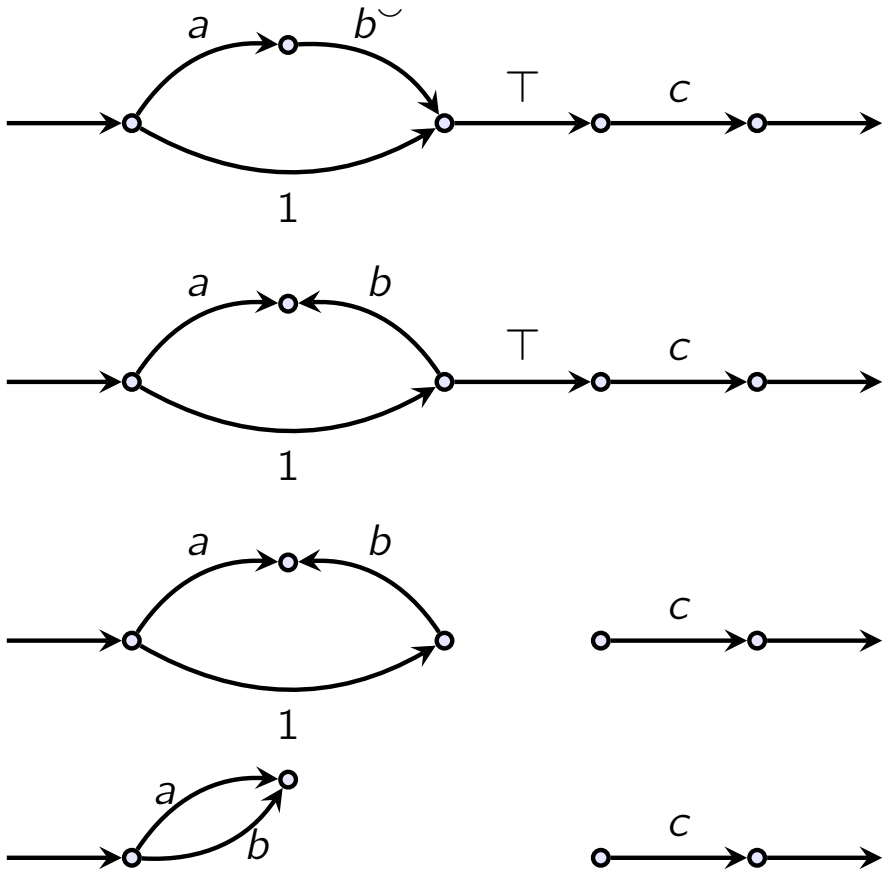

Lemma

$$\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\Phi(\mathcal{T}r(\mathcal{A})).$$

From $\mathrm{AReg}\langle\Sigma\rangle$ to $\mathrm{GReg}\langle\Sigma'\rangle$

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^\star \mid e^\smile \mid \top$$

$$e, f \in \mathrm{GReg}\langle\Sigma\rangle ::= 0 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^+$$

$$\Sigma' := \Sigma \cup \left\{ a^\smile \mid a \in \Sigma \right\} \cup \{1, \top\}.$$

$$\varphi : \mathrm{AReg}\langle\Sigma\rangle \to \mathrm{GReg}\langle\Sigma'\rangle$$

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$$e = (a \cdot b^\star)^\smile \cup \top$$

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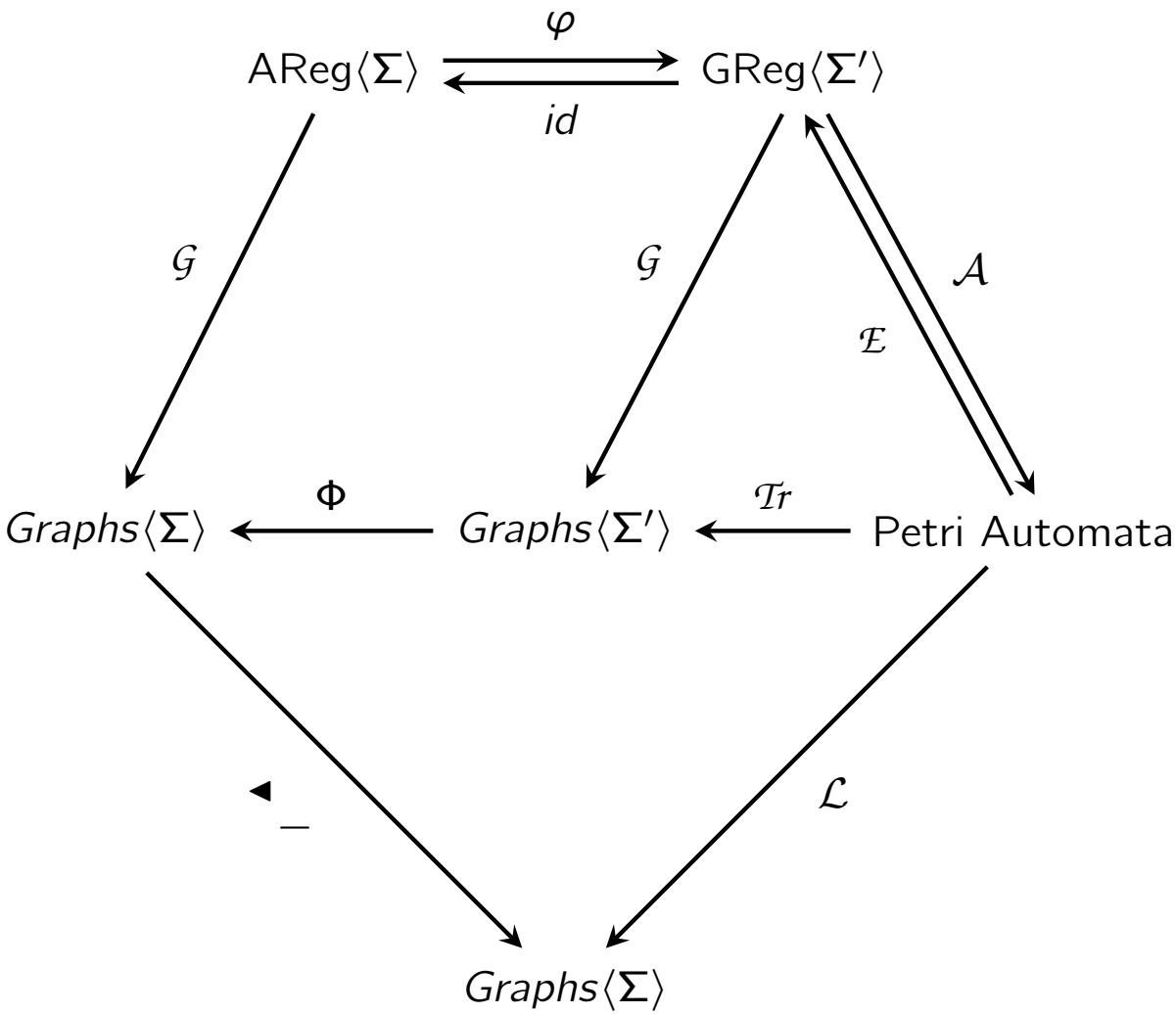
Lemma

$$\mathcal{G}_\Sigma(e) = \Phi\left(\mathcal{G}_{\Sigma'}(\varphi(e))\right).$$

Correspondences



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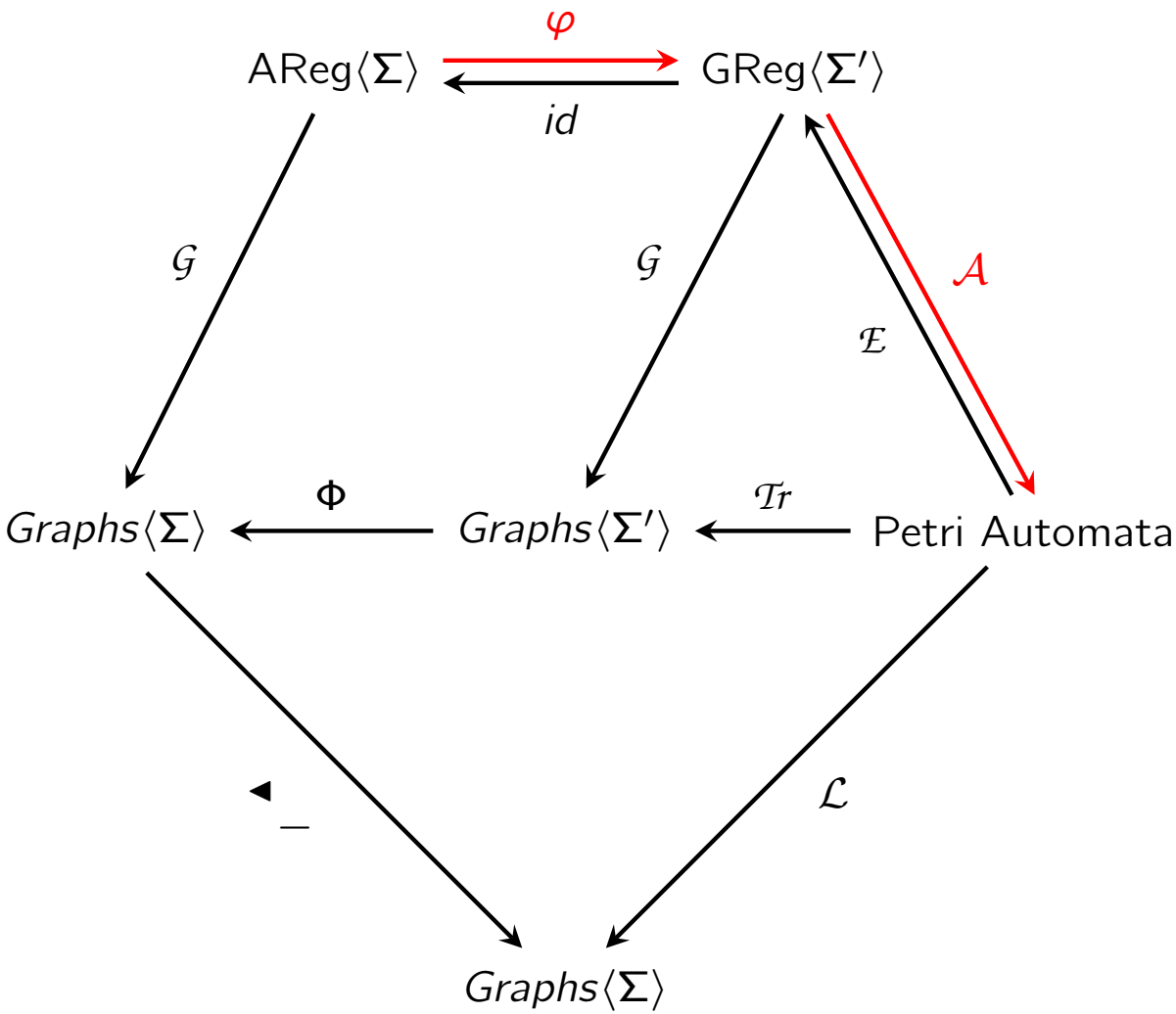

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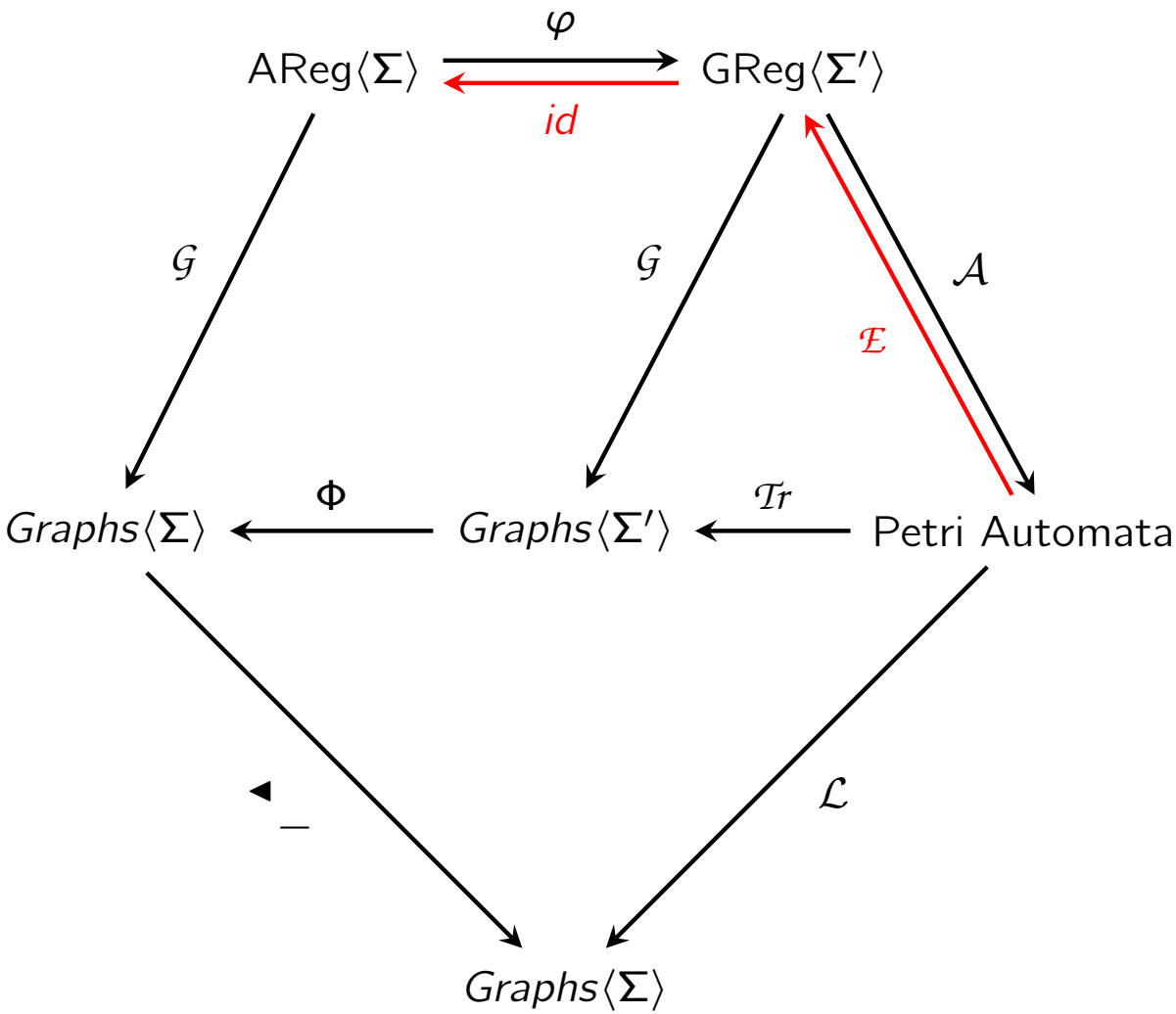

Kleene Theorem

Let $\mathcal{G}$ be a set of graphs labelled with Σ.

Theorem

$\mathcal{G}$ is the language recognised by a Petri automaton if and only if it is the closed graph language of an expression.

Corollary

The equational theory of bounded Kleene allegories is decidable if and only if the comparison of general Petri automata is decidable.

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- Kleene theorems for general Petri automata and $\mathrm{AReg}\langle\Sigma\rangle$ expressions.

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What am I doing these days?

- Concurrent Kleene Algebra.

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- Nominal Kleene Algebra.

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- Formalising results in `Coq`.

That's all folks!

Thank you!

See more at:
`http://paul.brunet-zamansky.fr`

Outline