Pomset languages and concurrent Kleene algebras

Completeness and decidability of CKA

Theory seminar, QMU – February 13, 2018

Paul Brunet[1], Damien Pous[2], Georg Struth[3], Tobias Kappé[1], Alexandra Silva[1], and Fabio Zanasi[1]

University College London[1], ENS de Lyon – CNRS[2], University of Sheffield[3]

UCL ENS DE LYON The University Of Sheffield.

Kleene Algebra

Equivalence of sequential programs

$$(x := 1; y := 2); (x := y \oplus y := x) \qquad \equiv \qquad x := 1; (y := 2; x := y) \oplus (y := 2; y := x)$$

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$$(x1 \cdot y2) \cdot (xy + yx) = x1 \cdot (y2 \cdot (xy + yx)) \qquad \text{(associativity of } \cdot\text{)}$$

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$$\begin{aligned}(x1 \cdot y2) \cdot (xy + yx) &= x1 \cdot (y2 \cdot (xy + yx)) && \text{(associativity of } \cdot\text{)} \\ &= x1 \cdot ((y2 \cdot xy) + (y2 \cdot yx)) && \text{(distributivity)}\end{aligned}$$

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A Kleene algebra is structure $\langle K, 0, 1, +, \cdot, \star \rangle$ such that:

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Theorem

$$\mathrm{KA} \vdash e = f \Leftrightarrow \mathcal{L}(e) = \mathcal{L}(f) .$$

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☞ Kozen, "A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events", 1991.

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Kleene Algebra

Equivalence of sequential programs

Kleene Algebra

$\Updownarrow$

regular languages

$\Updownarrow$

finite state automata

Kleene Algebra

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Completeness

Kleene Algebra

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Kleene Algebra

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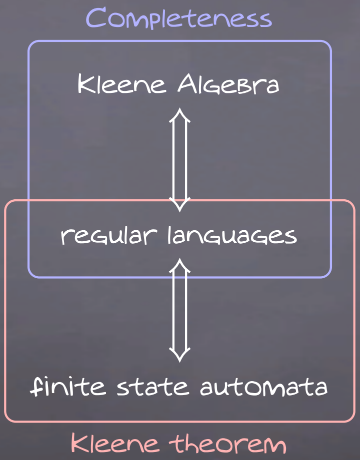
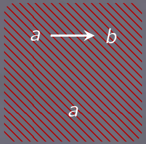


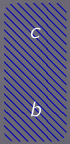
Concurrent Kleene Algebras
Equivalence of parallel programs

$$e, f \in \mathbb{E} ::= 0 \mid 1 \mid a \mid e + f \mid e \cdot f \mid e^\star \mid e \parallel f$$

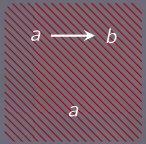
Bi-Kleene Algebra	Concurrent Kleene Algebra
$\Updownarrow$	$\Updownarrow$
series-rational pomset languages	down-closed series-rational languages
$\Updownarrow$	$\Updownarrow$
automata ?	automata ?

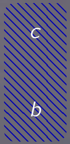
Pomsets

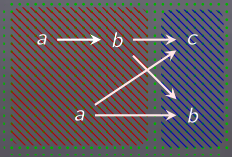
$P_1 =$ 

$P_2 =$ 

Pomsets

$P_1 =$ 

$P_2 =$ 

$P_1 \cdot P_2 =$ 

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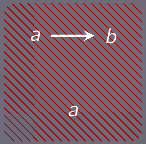

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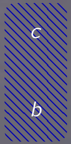

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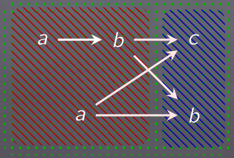
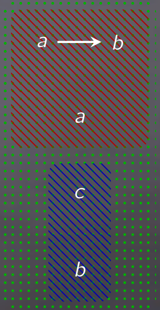


$P_1 \parallel P_2 =$



Rational pomset languages

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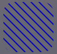
$[\![1]\!] :=$ {  }

$$[\![0]\!] := \emptyset$$

$$[\![e + f]\!] := [\![e]\!] \cup [\![f]\!]$$

$$[\![e \cdot f]\!] := [\![e]\!] \cdot [\![f]\!]$$

$$[\![e \parallel f]\!] := [\![e]\!] \parallel [\![f]\!]$$

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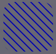
$[\![1]\!] := \{$  $\}$

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Definition

A set of pomsets S is called a rational pomset language if there is an expression $e \in \mathbb{E}$ such that $S = [\![e]\!]$.

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Theorem

$$\mathrm{biKA} \vdash e = f \Leftrightarrow [\![e]\!] = [\![f]\!].$$

☞ Laurence & Struth, "Completeness theorems for bi-Kleene algebras and series-parallel rational pomset languages", 2014.

Pomset order

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ c \arrow[r] \arrow[ur] & d \end{tikzcd} \quad \sqsubseteq \quad \begin{tikzcd} a \arrow[r] & b \\ c \arrow[r] & d \end{tikzcd}$$

Pomset order

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$$

Definition

$P_1 \sqsubseteq P_2$ if there is a function $\varphi : P_2 \to P_1$ such that:

1) φ is a bijection
2) φ preserves labels
3) φ preserves ordered pairs

☞ Gischer, "The equational theory of pomsets", 1988.

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Notation: ${}^{\sqsubseteq}S := \{P \mid \exists P' \in S : P \sqsubseteq P'\}$.

Concurrent Kleene algebra

A concurrent Kleene algebra is bi-Kleene algebra $\langle K, 0, 1, +, \cdot, \star, \parallel\rangle$ such that:

$$(a \parallel b) \cdot (c \parallel d) \leq (a \cdot c) \parallel (b \cdot d)$$

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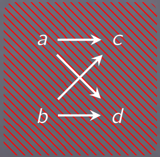
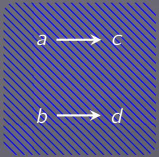


$\sqsubseteq$



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Theorem

$$\mathrm{CKA} \vdash e = f \Rightarrow {}^{\sqsubseteq}[\![e]\!] = {}^{\sqsubseteq}[\![f]\!].$$

☞ Hoare, Möller, Struth & Wehrman, "Concurrent Kleene Algebra", 2009.

Outline

$\mathbb{E}$

Outline

$$\begin{tikzcd} \mathbb{E} \arrow[d, "[\![-]\!]"] \\ \mathcal{P}(\mathbb{P}) \end{tikzcd}$$

Outline

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "\llbracket \_ \rrbracket"] \\
\mathcal{P}(\mathbb{P}) \arrow[d, "{}^{\sqsubseteq}(\_)"] \\
\mathcal{P}(\mathbb{P})
\end{tikzcd}
$$

Outline

$$\begin{tikzcd}
 & \mathbb{E} \arrow[dl, bend right, "(_)\downarrow"'] \arrow[d, "[\![_]\!]"] \\
\mathbb{E} & \mathcal{P}(\mathbb{P}) \arrow[d, "{}^{\sqsubseteq}(_)"] \\
 & \mathcal{P}(\mathbb{P})
\end{tikzcd}$$

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$$
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Outline

1. Completeness

$$
\begin{tikzcd}
& \mathbb{E} \arrow[dl, bend right, "(\_)\downarrow"'] \arrow[d, "[\![\_]\!]"] \\
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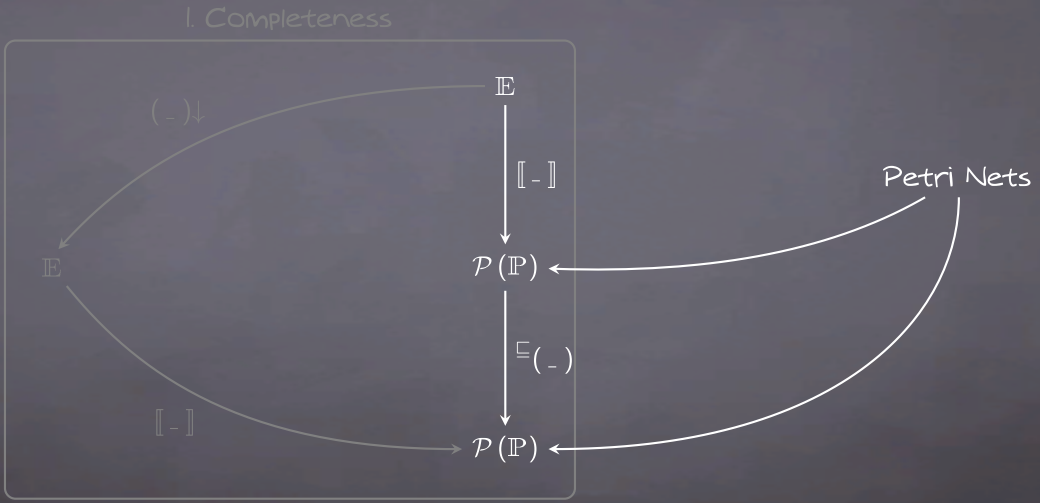
1. Completeness
$\mathbb{E}$
$(\_)\downarrow$
$[\![\_]\!]$
$\mathbb{E}$
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${}^{\sqsubseteq}(\_)$
$[\![\_]\!]$
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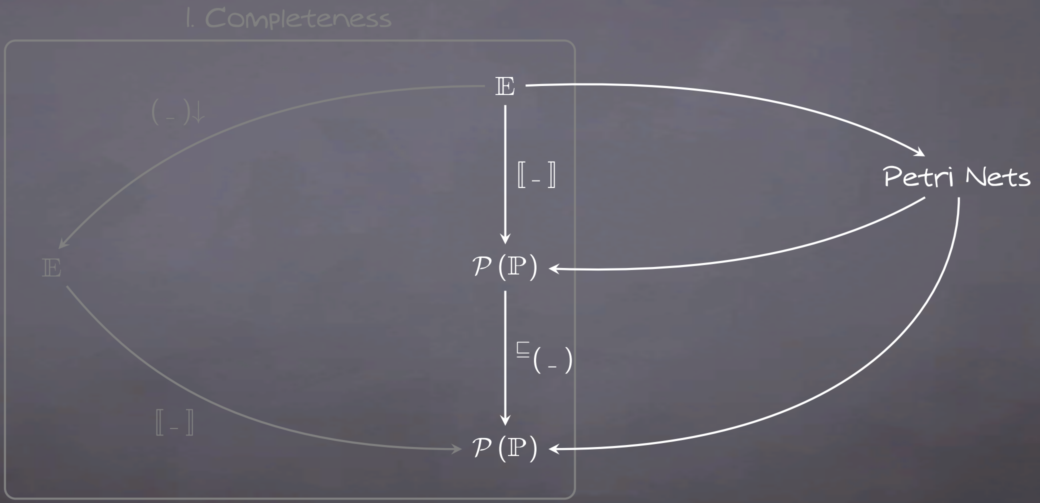
Petri Nets

Outline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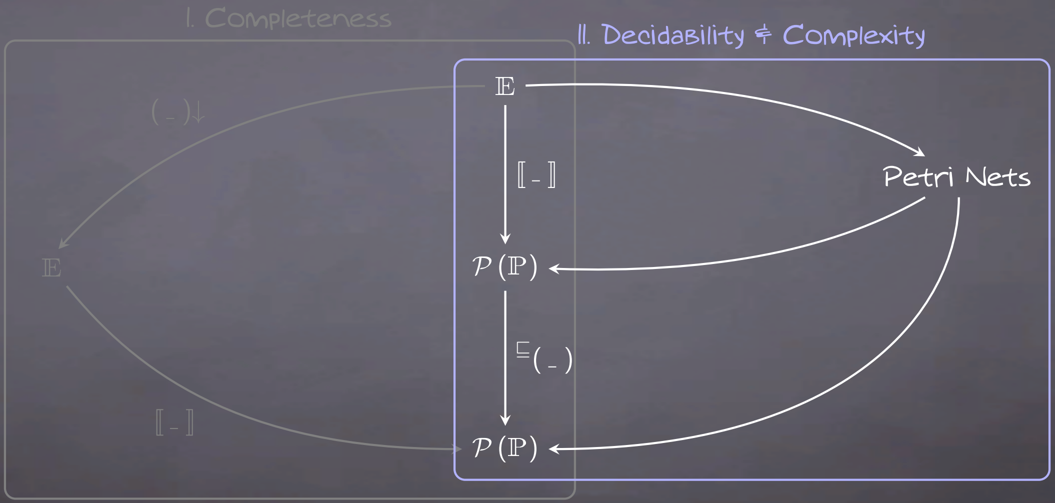
Outline


I. Completeness
II. Decidability & Complexity
$\mathbb{E}$
$(\_)\!\downarrow$
$\mathbb{E}$
$[\![\_]\!]$
$[\![\_]\!]$
$\mathcal{P}(\mathbb{P})$
${}^{\sqsubseteq}(\_)$
$\mathcal{P}(\mathbb{P})$
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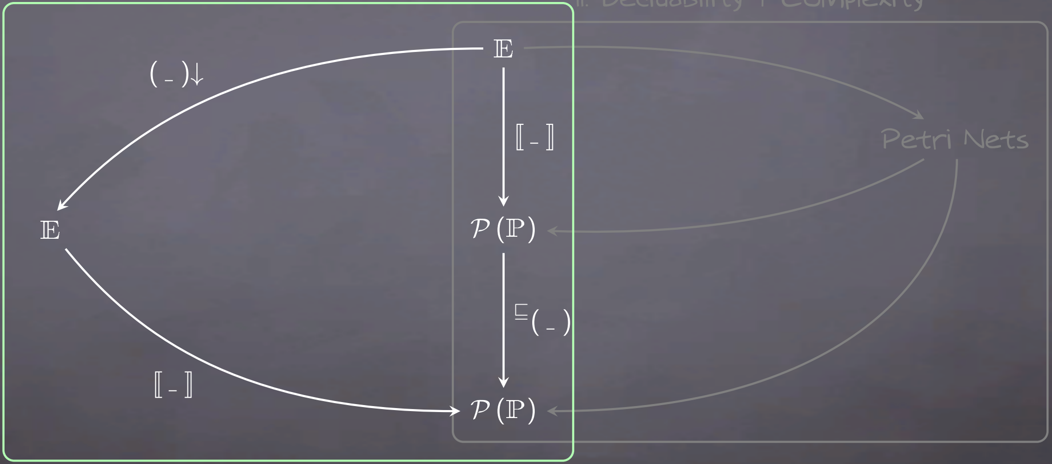
Outline

Kappé, Brunet, Silva & Zanasi, "Concurrent Kleene Algebra: Free Model and Completeness", 2018

I. Completeness

II. Decidability & Complexity



Syntactic closures are nice...

Definition

An expression $e{\downarrow}$ is a closure of e if $\mathrm{CKA} \vdash e{\downarrow} = e$ and $[\![e{\downarrow}]\!] = {}^{\sqsubseteq}[\![e]\!]$.

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Lemma

If every series-rational expression admits a closure, the axioms of CKA are complete with respect to down-closed pomset languages.

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By completeness of biKA, it follows that $\mathrm{biKA} \vdash e{\downarrow} = f{\downarrow}$, thus $\mathrm{CKA} \vdash e{\downarrow} = f{\downarrow}$.

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1) every <u>strict subterm</u> of $e \parallel f$,
2) every term with <u>smaller width</u> than $e \parallel f$.

... but do they exist?

Let's try and compute the closure by induction:

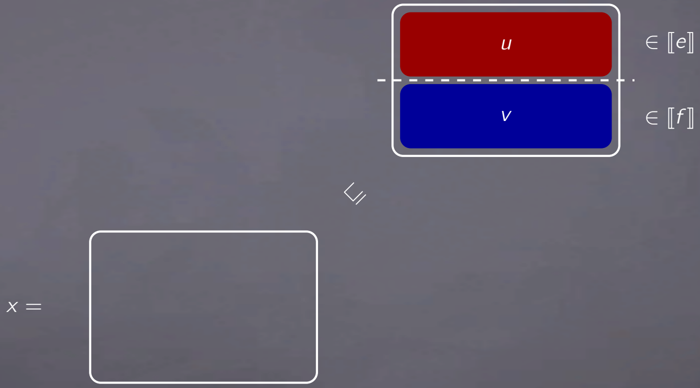
- $0{\downarrow} = 0$
- $1{\downarrow} = 1$
- $a{\downarrow} = a$
- $(e + f){\downarrow} = e{\downarrow} + f{\downarrow}$
- $(e \cdot f){\downarrow} = e{\downarrow} \cdot f{\downarrow}$
- $(e^\star){\downarrow} = e{\downarrow}^\star$
- $(e \parallel f){\downarrow} = ???$

We strengthen our induction, by assuming that we have closures for

1) every strict subterm of $e \parallel f$,
2) every term with smaller width than $e \parallel f$.

We write the corresponding strict ordering $\prec$.

Who's smaller than a parallel product?



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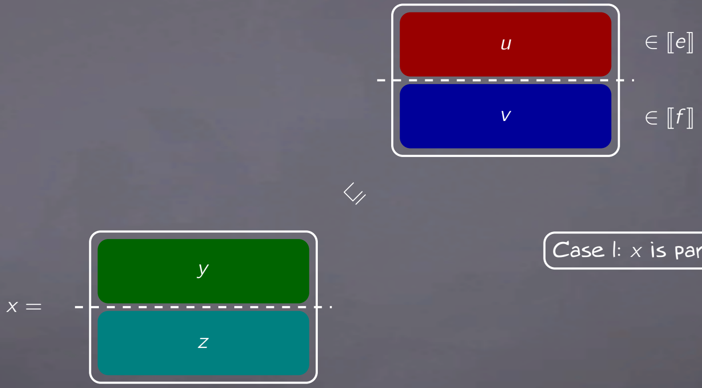
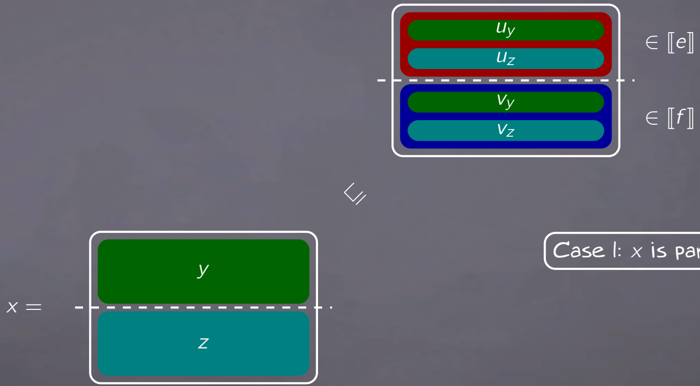


Case 1: x is parallel.

Who's smaller than a parallel product?



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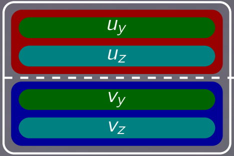
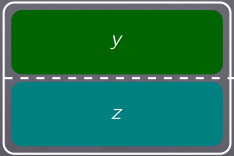


$\in [\![e]\!]$

$\in [\![f]\!]$

$\sqsubseteq_{\parallel}$

$x =$



$\sqsubseteq u_y \parallel v_y$

$\sqsubseteq u_z \parallel v_z$

Case 1: x is parallel.

Parallel splicing and preclosure

Parallel splicing

Δ_e is a finite relation over $\mathbb{E}$ such that:

$$u \parallel v \in [\![e]\!] \Leftrightarrow \exists l \; \Delta_e \; r : u \in [\![l]\!] \wedge v \in [\![r]\!].$$

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$$e \odot f = e \parallel f + \sum_{l \Delta_{e \parallel f} r} (l{\downarrow}) \parallel (r{\downarrow}) \,.$$

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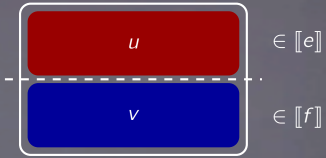
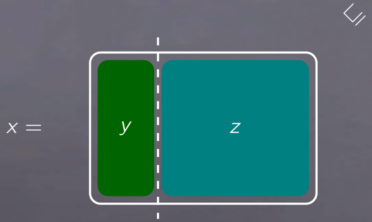
Lemma

$$u \parallel v \in {}^{\sqsubseteq}[\![e \parallel f]\!] \Leftrightarrow u \parallel v \in [\![e \odot f]\!].$$

$$\mathrm{CKA} \vdash e \odot f = e \parallel f.$$

Who's smaller than a parallel product?

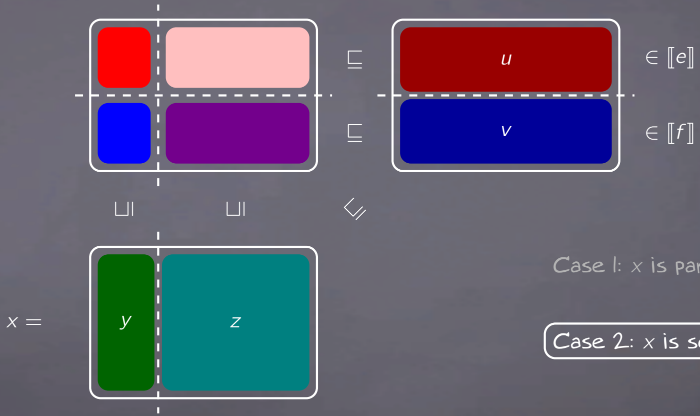




Case 1: x is parallel.

$\boxed{\text{Case 2: } x \text{ is sequential.}}$

Who's smaller than a parallel product?



Case 1: x is parallel.

$\boxed{\text{Case 2: } x \text{ is sequential.}}$

Sequential splicing

Sequential splicing

∇_e is a finite relation over $\mathbb{E}$ such that:

$$u \cdot v \in [\![e]\!] \Leftrightarrow \exists l \; \nabla_e \; r : u \in [\![l]\!] \wedge v \in [\![r]\!].$$

Sequential splicing

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∇_e is a finite relation over $\mathbb{E}$ such that:

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$$u \cdot v \in {}^{\sqsubseteq}[\![e \parallel f]\!] \Leftrightarrow u \cdot v \in [\![e \parallel f + \sum_{\substack{l_e \ \nabla_e \ r_e \\ l_f \ \nabla_f \ r_f}} (l_e \odot l_f) \cdot (r_e \parallel r_f){\downarrow}]\!]$$

Sequential splicing

Sequential splicing

∇_e is a finite relation over $\mathbb{E}$ such that:

$$u \cdot v \in [\![e]\!] \Leftrightarrow \exists l \nabla_e r : u \in [\![l]\!] \wedge v \in [\![r]\!].$$

$$u \cdot v \in {}^{\sqsubseteq}[\![e \parallel f]\!] \Leftrightarrow u \cdot v \in [\![e \parallel f + \sum_{\substack{l_e \nabla_e r_e \\ l_f \nabla_f r_f}} (l_e \odot l_f) \cdot (r_e \parallel r_f)\downarrow]\!]$$

Problem: $r_e \parallel r_f$ is not always smaller than $e \parallel f$...

And then, some magic happens...

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☞ We repeat the construction to get successive equations, involving closures.

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- Only a finite number of unknown closures appear.

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And then, some magic happens...

- We repeat the construction to get successive equations, involving closures.
- Only a finite number of unknown closures appear.
- These equations can be structured as a linear system.
- With a fancy fixpoint theorem, we compute the least solution of the system.
- This solution is a closure.

Completeness of CKA

Lemma

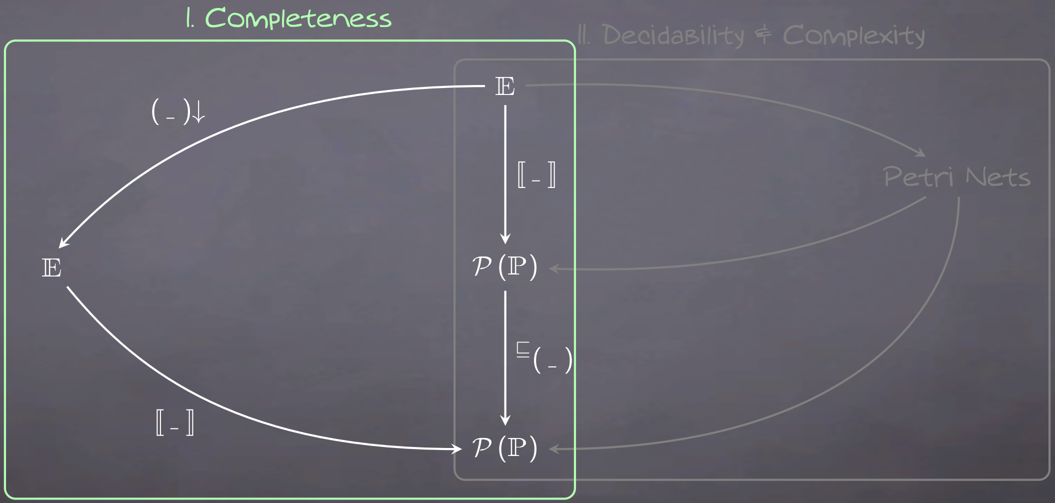
Every series-rational expression admits a closure.

Theorem

$$\mathrm{CKA} \vdash e = f \Leftrightarrow {}^{\sqsubseteq}[\![e]\!] = {}^{\sqsubseteq}[\![f]\!].$$

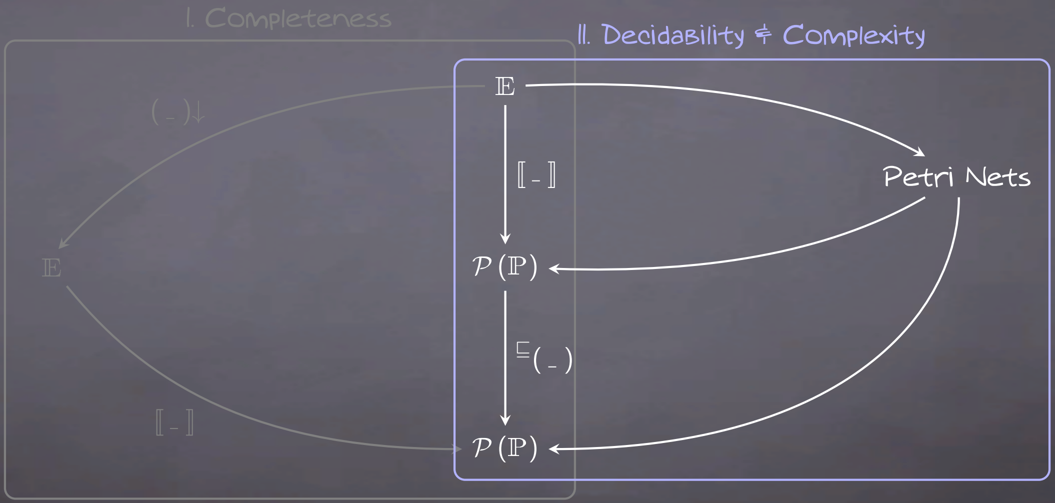
Implementation: `https://doi.org/10.5281/zenodo.926651`.

Outline


I. Completeness
II. Decidability & Complexity
$\mathbb{E}$
$(\_)\downarrow$
$[\![\_]\!]$
$\mathbb{E}$
$\mathcal{P}(\mathbb{P})$
${}^{\sqsubseteq}(\_)$
$[\![\_]\!]$
$\mathcal{P}(\mathbb{P})$
Petri Nets

Outline

Brunet, Pous & Struth, "On decidability of concurrent Kleene algebra", 2017



Two decision problems

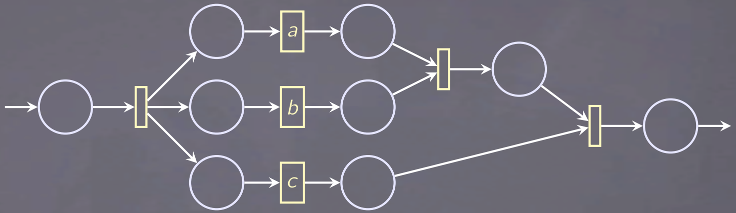
biKA

Given two expressions e, f, are $[\![e]\!]$ and $[\![f]\!]$ equal?

CKA

Given two expressions e, f, are ${}^{\sqsubseteq}[\![e]\!]$ and ${}^{\sqsubseteq}[\![f]\!]$ equal?

Labelled Petri nets



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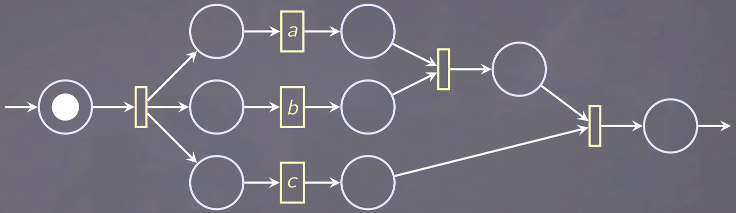
Labelled Petri nets



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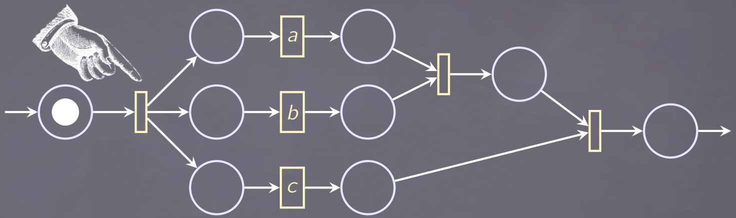

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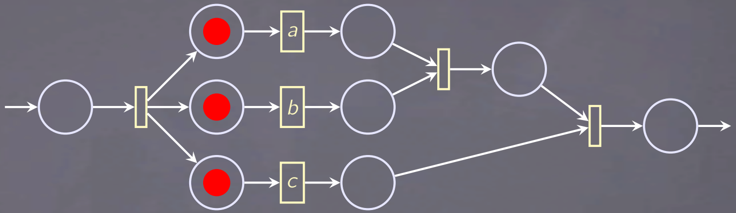

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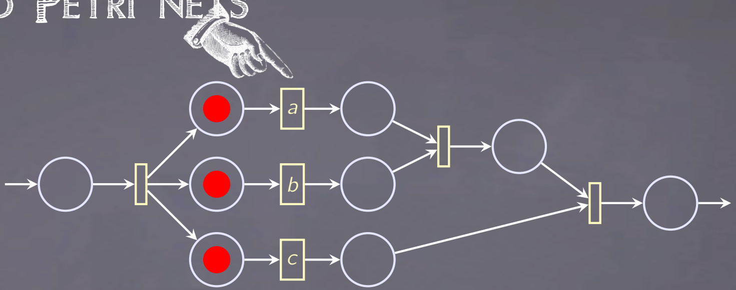

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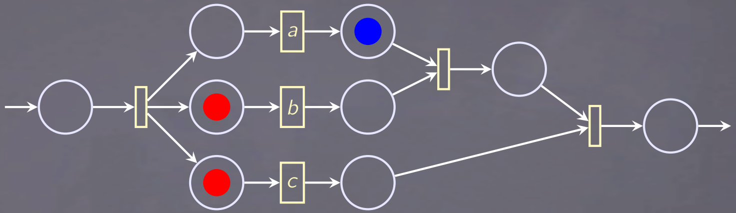

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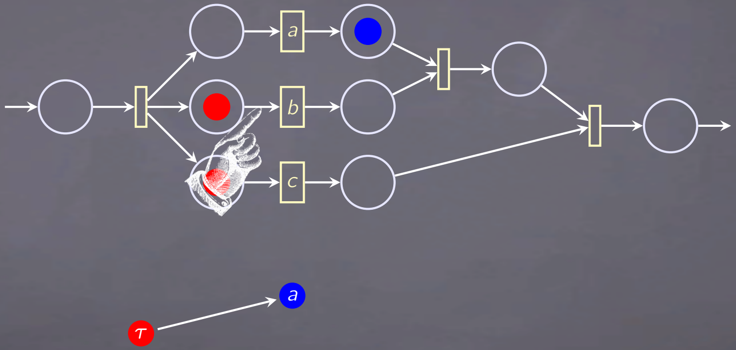

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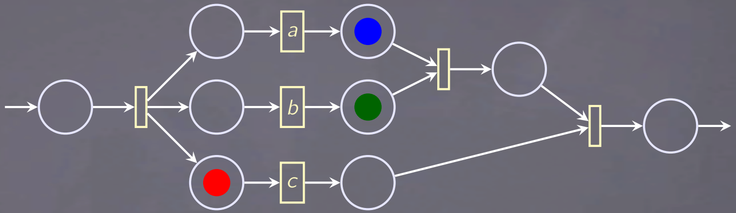

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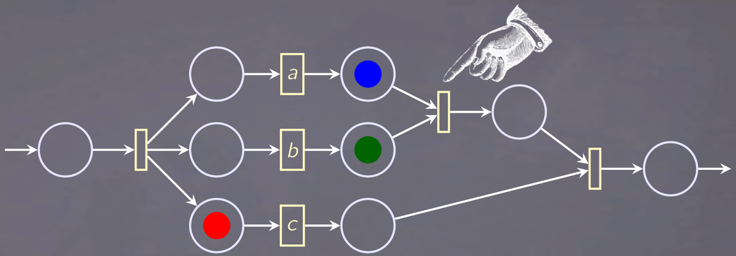

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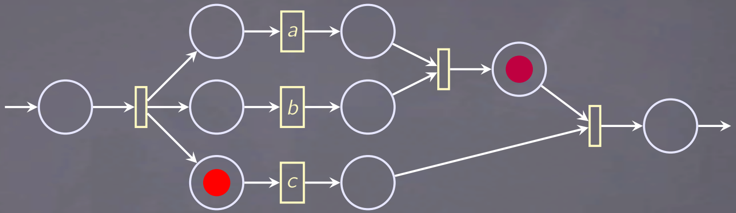

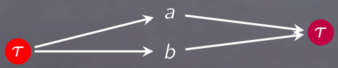

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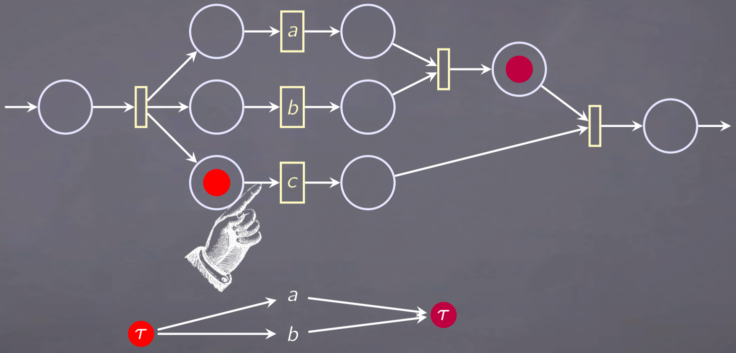

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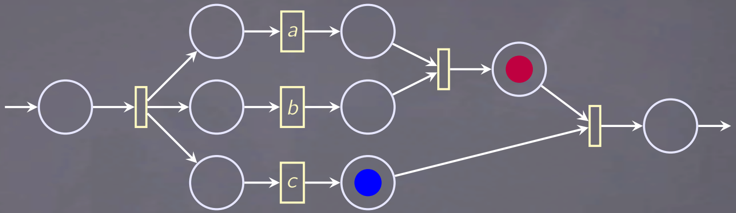

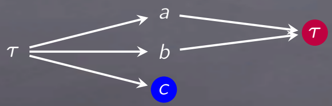

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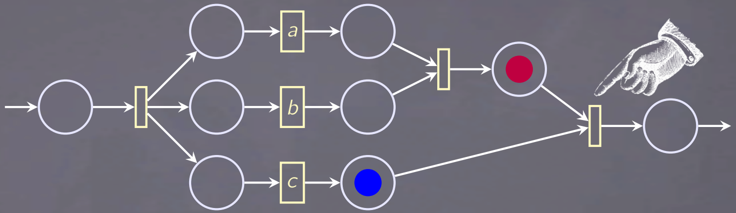

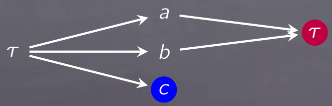

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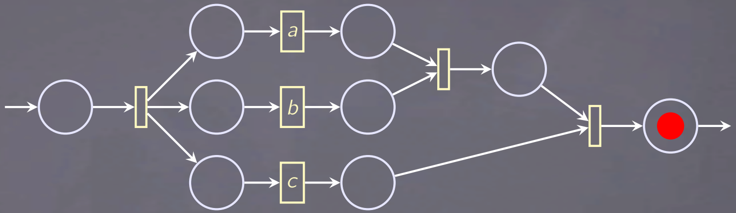

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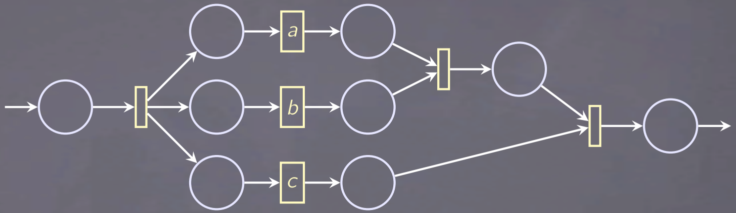

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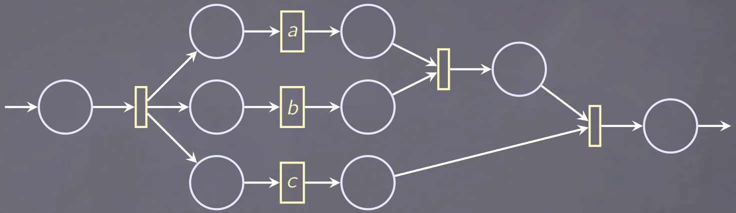

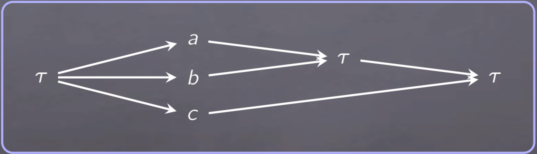

Transition-pomset

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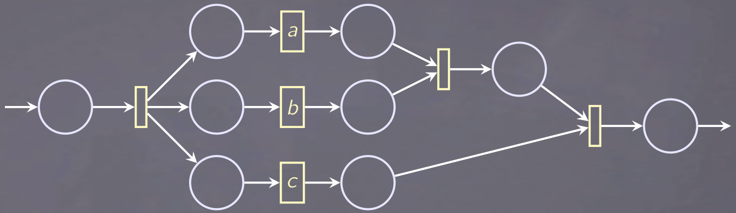
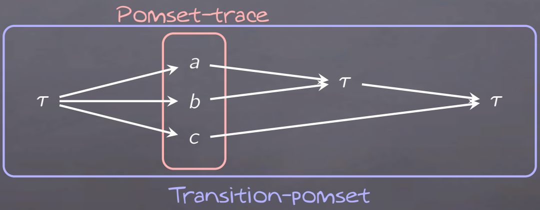
Labelled Petri nets





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Recognisable pomset languages

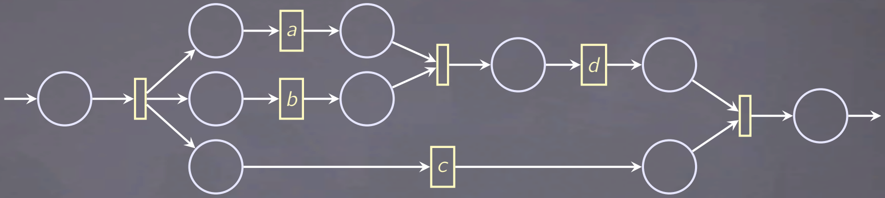
Language generated by a net

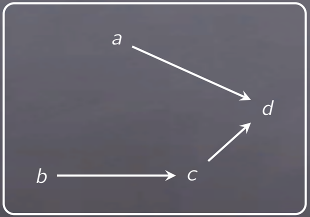
$[\![\mathcal{N}]\!]$ is the set of pomset-traces of accepting runs of $\mathcal{N}$.

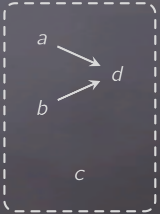
Definition

A set of pomsets S is a recognisable pomset language if there is a net $\mathcal{N}$ such that $S = [\![\mathcal{N}]\!]$.

Reading a pomset in a net







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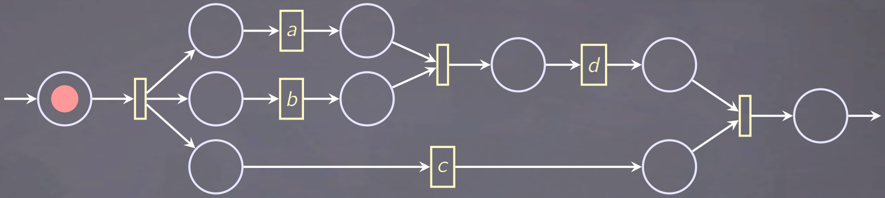
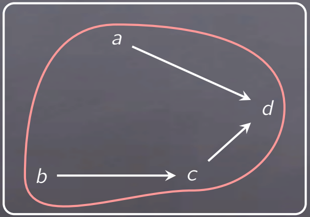
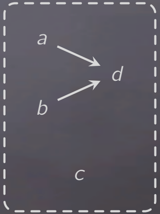
Reading a pomset in a net







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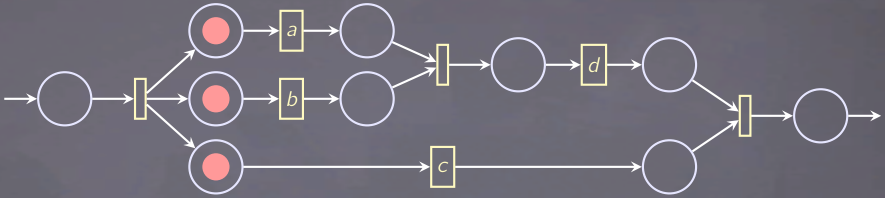

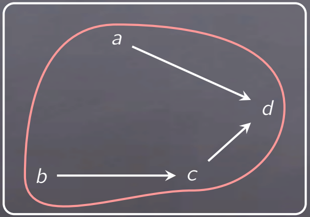

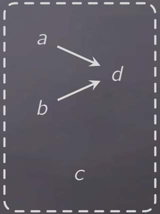

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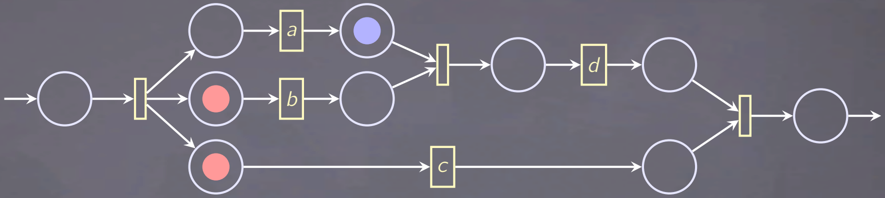

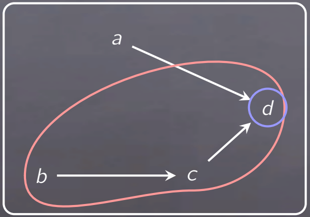

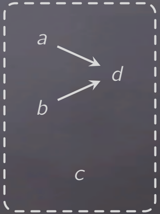

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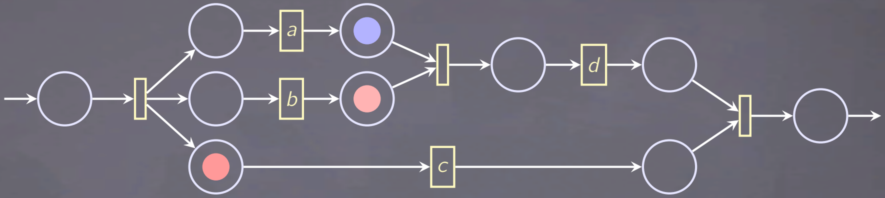

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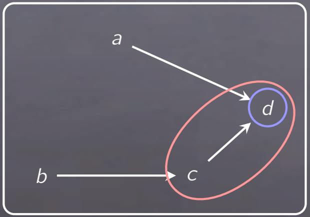

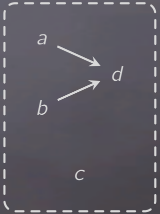

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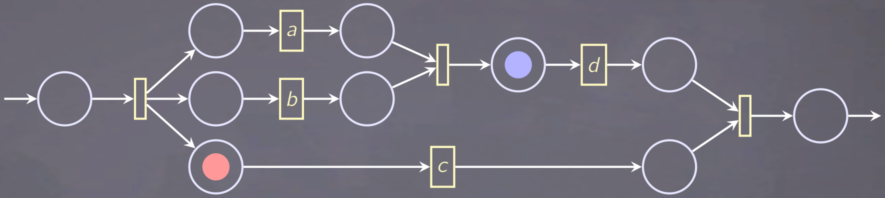

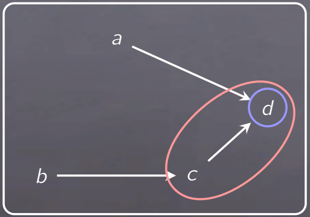

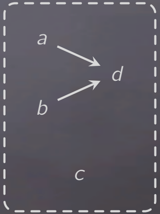

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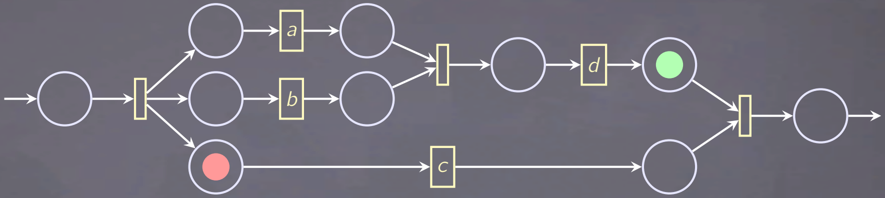

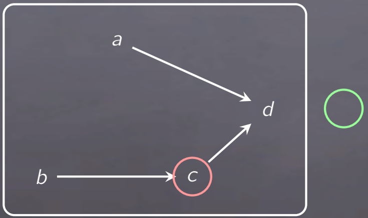

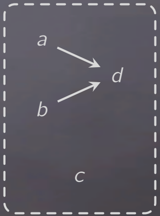

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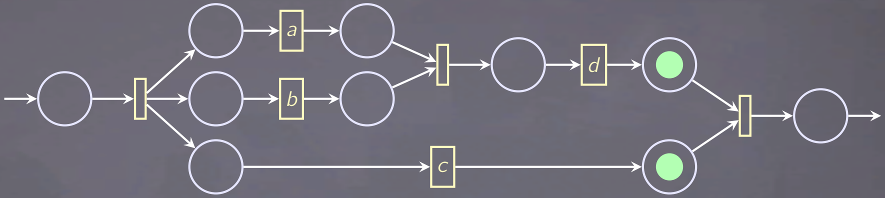

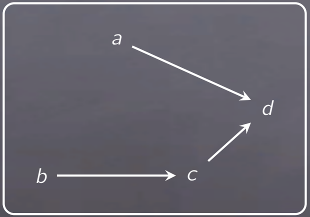

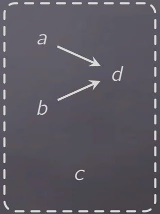

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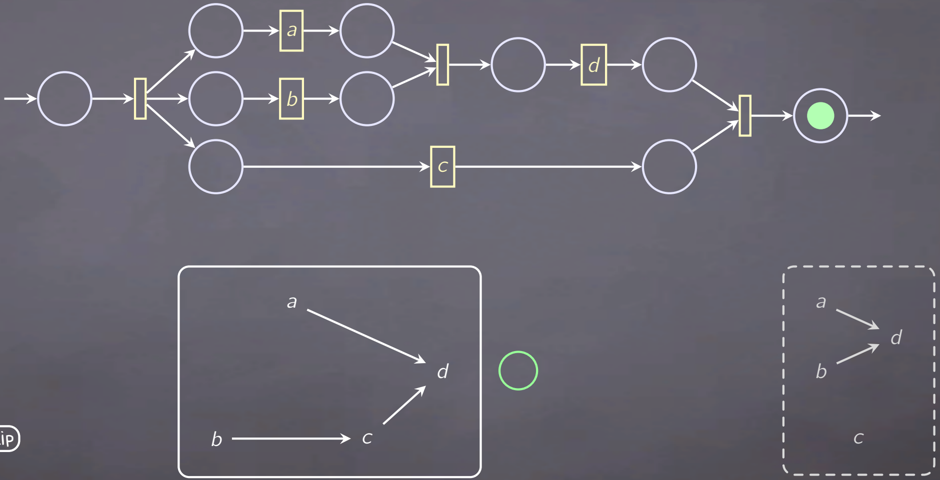

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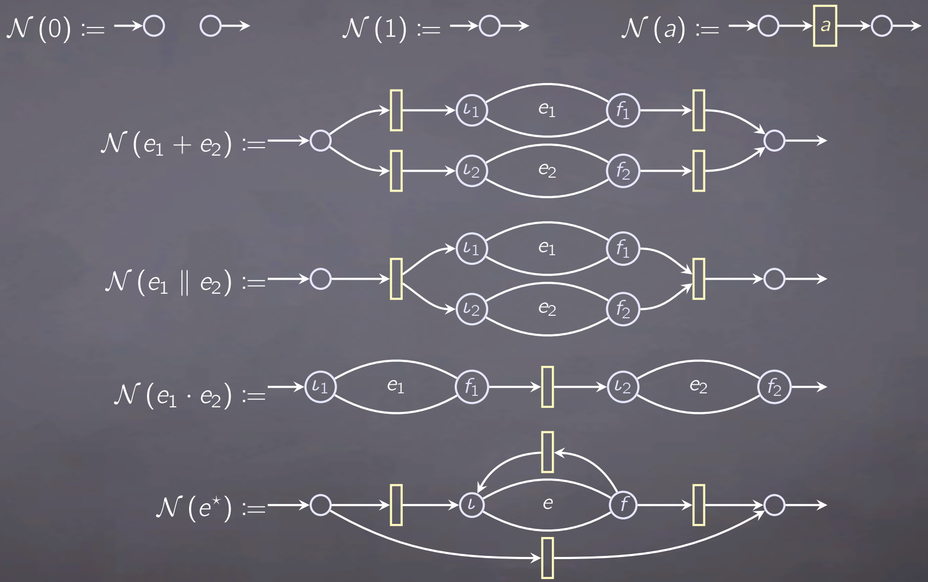
From expressions to automata



Solving biKA

Lemma

$$[\![e]\!] = [\![\mathcal{N}(e)]\!].$$

Corollary: Rational pomset languages are recognisable.

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Testing containment of pomset-trace languages of two Petri nets is an ExpSpace-complete problem.

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Corollary: The problem biKA lies in the class ExpSpace.

What about CKA?

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Problem

Let $\mathcal{N}_1, \mathcal{N}_2$ be well behaved nets. Is it true that for every run R_1 of $\mathcal{N}_1$ there is a run R_2 in $\mathcal{N}_2$ such that

$$\mathcal{P}om(R_1) \sqsubseteq \mathcal{P}om(R_2)?$$

Idea of the algorithm

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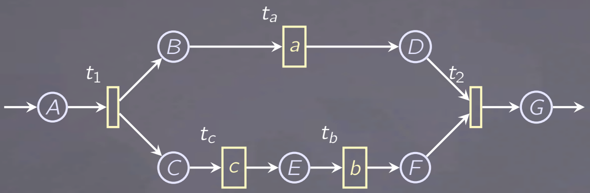
- ☞ $[\![\mathcal{N}_1]\!] \subseteq^{\sqsubseteq} [\![\mathcal{N}_2]\!]$ if and only if $\mathcal{L}(\mathscr{A}_1) = \mathcal{L}(\mathscr{A}_2)$.

Transition automaton



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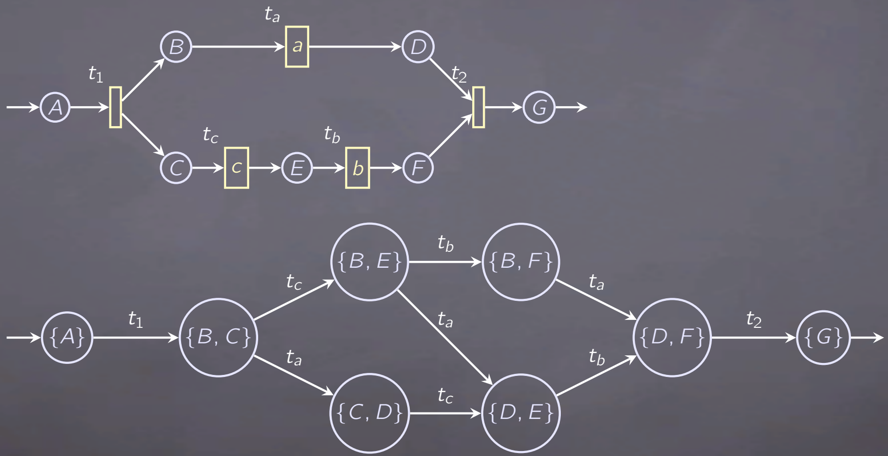

Massaging runs

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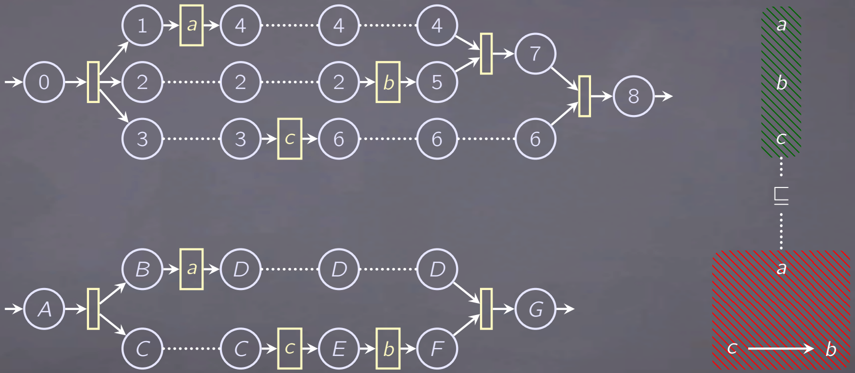

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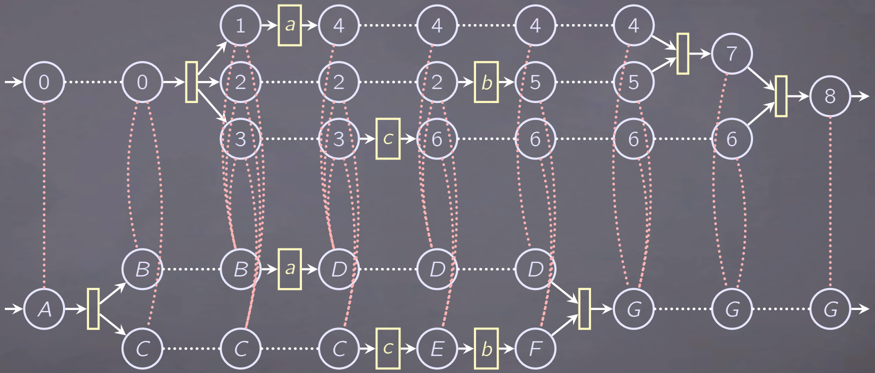

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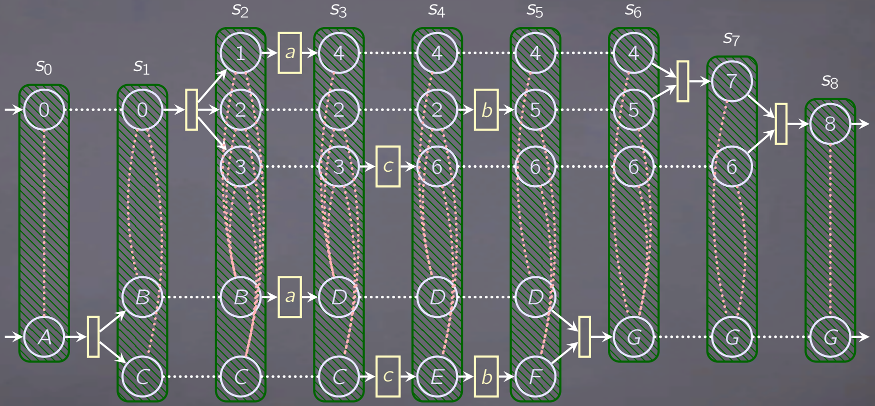

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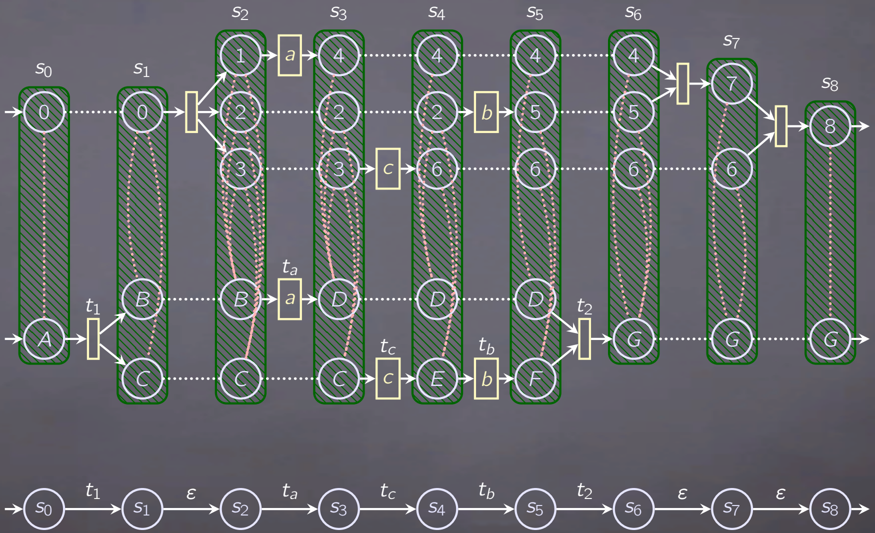

Reduction to automata

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be some polite nets, of size n, m.

Lemma

There is an automaton $\mathscr{A}(\mathcal{N}_1)$ with $\mathcal{O}(2^n)$ states that recognises the set of accepting runs in $\mathcal{N}_1$.

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Lemma

There is an automaton $\mathcal{N}_1 \prec \mathcal{N}_2$ with $\mathcal{O}(2^{n+m+nm})$ states that recognises the set of accepting runs in $\mathcal{N}_1$ whose pomset belongs to ${}^{\sqsubseteq}[\![\mathcal{N}_2]\!]$.

Decidability + Complexity

Theorem

Given two expressions $e, f \in \mathbb{E}$, we can test if $[\![e]\!] \subseteq^{\sqsubseteq} [\![f]\!]$ in ExpSpace.

Proof.

1) Build $\mathcal{N}(e)$ and $\mathcal{N}(f)$;
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3) compare them.

$\square$

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The problem CKA is ExpSpace-hard.

(Universality problem for regular expressions with interleaving)

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Corollary: The problem CKA is ExpSpace-complete.

Outline

$\mathbb{E}$

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$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "[\![\_]\!]"] \\
\mathcal{P}(\mathbb{P})
\end{tikzcd}
$$

Outline

$$\begin{tikzcd} \mathbb{E} \arrow[d, "[\![\_]\!]"] \\ \mathcal{P}(\mathbb{P}) \arrow[d, "{}^{\sqsubseteq}(\_)"] \\ \mathcal{P}(\mathbb{P}) \end{tikzcd}$$

Outline

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\begin{tikzcd}
 & \mathbb{E} \arrow[dl, bend right, "(\_)\!\downarrow"'] \arrow[d, "\llbracket \_ \rrbracket"] \\
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Outline

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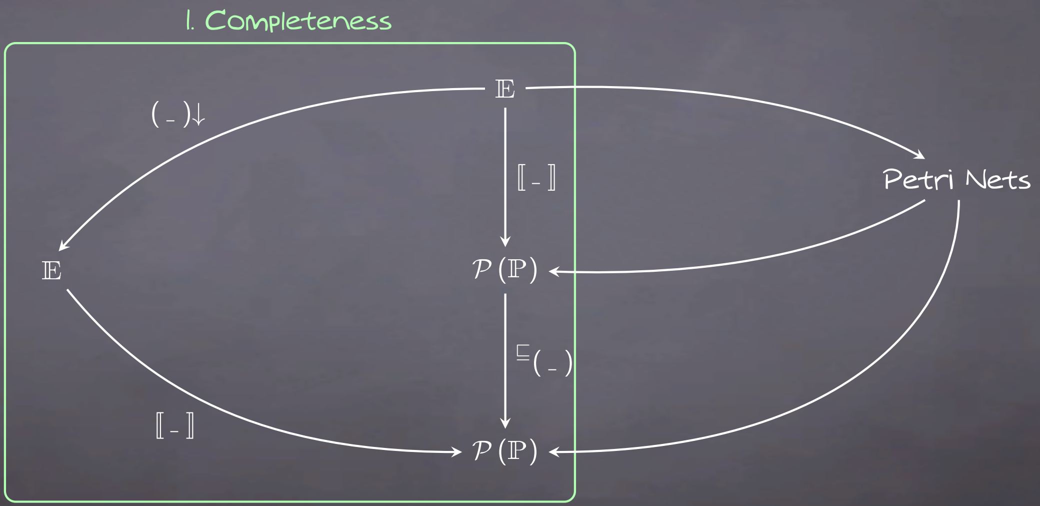
\mathbb{E} \arrow[dr, bend right, "{[\![_]\!]}"'] & \mathcal{P}(\mathbb{P}) \arrow[d, "{}^{\sqsubseteq}(_)"] & \text{Petri Nets} \arrow[l, bend left] \arrow[dl, bend left] \\
 & \mathcal{P}(\mathbb{P}) &
\end{tikzcd}$$

Outline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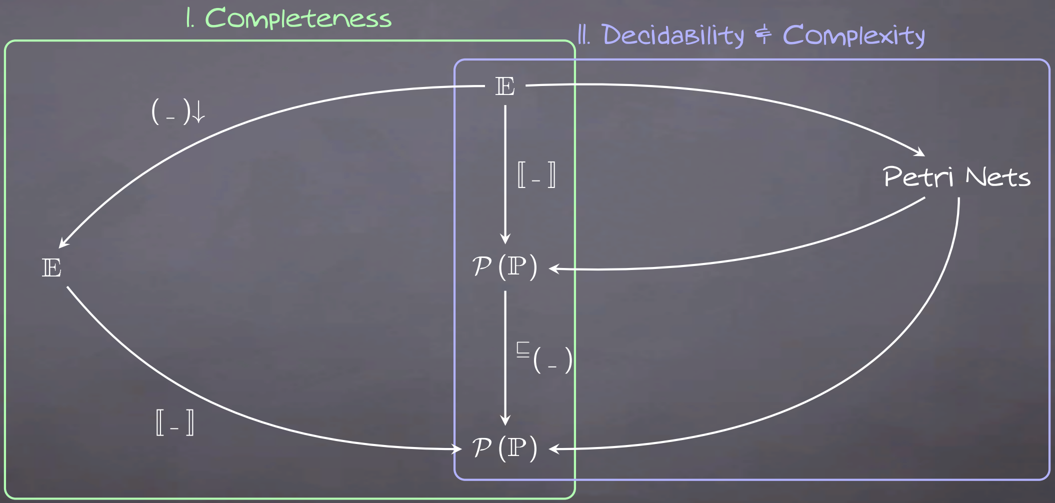
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- ☞ Insert you favourite operator here...

That's all folks!

Thank you!

See more at:
http://paul.brunet-zamansky.fr

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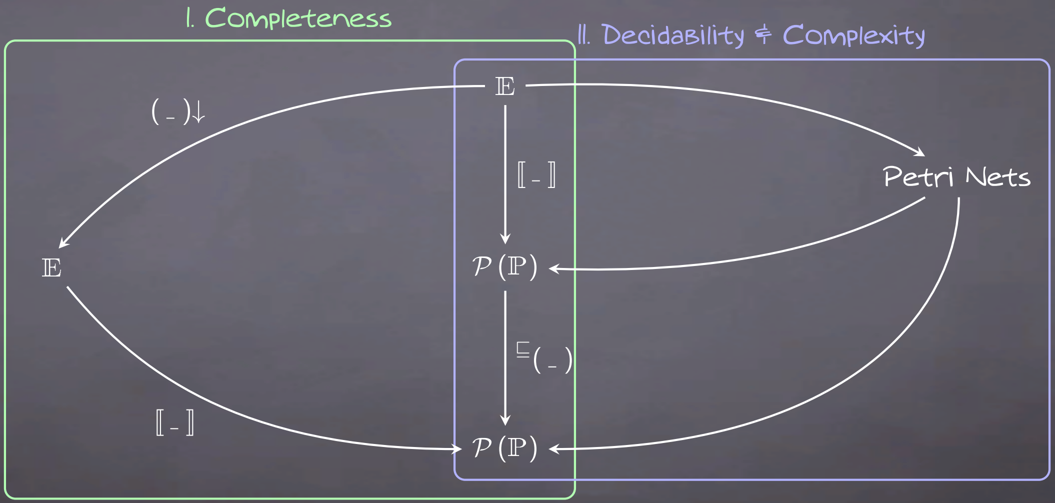
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I. Introduction

II. Completeness

III. Decidability & Complexity

IV. Summary and Outlook