On preorders

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Notations *A binary relation* $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$ *is a subset of* $\mathrm{A} \times \mathrm{B}$. *We denote the set-theoretic inclusion of relations by* $\rightarrowtriangle$, *writing* $x \rightarrowtriangle y$ *instead of* $x \subseteq y$. *Similarly, equality of relations is denoted by* $\leftrightarrowtriangle$. *Given sets* $\mathrm{A}, \mathrm{B}, \mathrm{C}$, *and relations* $x, x' : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$ *and* $y : \mathrm{B} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{C}$, *we define the following constants and operations:* $1_{\mathrm{A}} : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ *is the identity (diagonal) relation over* A, $x^{\smile} : \mathrm{B} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ *is the converse of* x, $x + x' : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$ *is the set-theoretic union of* x *and* x', *and* $x \mathbin{;} y : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{C}$ *is the sequential composition of* x *and* y, *containing all pairs* (a, c) *such that there is some* $b \in \mathrm{B}$ *such that* $(a, b) \in x$ *and* $(b, c) \in y$. *If* $s : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ *is a relation, its reflexive-transitive closure is written* $s^{\star}$. *Given* $z : \mathrm{A} \rightarrow \mathrm{C}$, *the (left-)residual of* z *by* x, *written* $x \setminus z : \mathrm{B} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{C}$, *is the set of pairs* (b, c) *such that for every* $a \in \mathrm{A}$, $(a, b) \in x$ *entails* $(a, c) \in z$. *This operator has can also be defined by the following equivalence:*

$$\forall x, y, z,\ y \rightarrowtriangle x \setminus z \Leftrightarrow x \mathbin{;} y \rightarrowtriangle z. \tag{1}$$

A dual operator $z \mathbin{/} y$, *satisfying* $x \rightarrowtriangle z \mathbin{/} y$ *iff* $x \mathbin{;} y \rightarrowtriangle z$ *can be defined by* $z \mathbin{/} y := ((y^{\smile}) \setminus (z^{\smile}))^{\smile}$.

A preorder is a reflexive and transitive relation $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$, over some set A. In relation algebra, this is captured by the following axioms:

$$1_{\mathrm{A}} \rightarrowtriangle x \tag{reflexivity}$$

$$x \mathbin{;} x \rightarrowtriangle x \tag{transitivity}$$

The class of preorders is stable by converse, and for any relation $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$, both $x \setminus x$ and $x \mathbin{/} x$ are preorders, respectively over B and A. We may further observe that for any preorder $y : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$, the relation y is equivalent to $y \setminus y$ (and to $y \mathbin{/} y$). Therefore this may be used as an alternative axiomatisation of preorders: a relation x is a preorder iff $x \leftrightarrowtriangle x \setminus x$.

A similar axiomatisation already exists: preorders may be defined in terms of reflexive transitive closure: a relation $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ is a preorder iff $x \leftrightarrowtriangle x^{\star}$. Therefore, in the case of preorders, $x^{\star} \leftrightarrowtriangle x \setminus x \leftrightarrowtriangle x \mathbin{/} x$. This is remarkable, given that for arbitrary relations $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$, the preorders $x \setminus x$ and $x \mathbin{/} x$ have different types, and $x^{\star}$ is undefined unless $\mathrm{A} = \mathrm{B}$.

Consider now the question of composing preorders: is the result a preorder itself? It turns out this question is equivalent to a form of commutativity between the orders. Indeed, the composite $x \mathbin{;} y$ of two preorders x, y over the same set A is a preorder if and only if $y \mathbin{;} x \rightarrowtriangle x \mathbin{;} y$. Futher, we can prove that it is also equivalent to the following identity:

$$x \mathbin{;} y \leftrightarrowtriangle (x + y)^{\star}.$$

Proofs

Lemma 1. *The converse of a preorder is a preorder.*

Proof. Let $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ be some preorder. We check that $x^{\smile}$ satisfies the two axioms:

- reflexivity: since $1_{\mathrm{A}} \rightarrowtriangle x$ and $1_{\mathrm{A}}^{\smile} \leftrightarrowtriangle 1_{\mathrm{A}}$, we get by monotonicity of $-^{\smile}$ that

$$1_{\mathrm{A}} \leftrightarrowtriangle 1_{\mathrm{A}}^{\smile} \rightarrowtriangle x^{\smile}.$$

- transitivity: since $x \mathbin{;} x \rightarrowtriangle x$ by monotonicity of $^{\smile}$ and the fact that for any relations y, z we have $y^{\smile} \mathbin{;} z^{\smile} \leftrightarrowtriangle (z \mathbin{;} y)^{\smile}$, we get:

$$x^{\smile} \mathbin{;} x^{\smile} \leftrightarrowtriangle (x \mathbin{;} x)^{\smile} \rightarrowtriangle x^{\smile}.$$ □

Preorders and residuals

Lemma 2. *For any relation $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{B}$, $x \setminus x$ and $x \mathbin{/} x$ are preorders.*

Proof. We check both axioms:

- reflexivity of $x \setminus x$: $1 \rightarrowtriangle x \setminus x \Leftrightarrow x \rightarrowtriangle x$
- transitivity of $x \setminus x$: $(x \setminus x) \mathbin{;} (x \setminus x) \rightarrowtriangle x \setminus x \Leftrightarrow x \mathbin{;} (x \setminus x) \mathbin{;} (x \setminus x) \rightarrowtriangle x$

So for any relation x, $x \setminus x$ is a preorder, meaning in particular $x^{\smile} \setminus x^{\smile}$ is a preorder. Since the converse of a preorder is a preorder, the relation

$$(x^{\smile} \setminus x^{\smile})^{\smile} \leftrightarrowtriangle (x^{\smile})^{\smile} \mathbin{/} (x^{\smile})^{\smile} \leftrightarrowtriangle x \mathbin{/} x$$

is a preorder. □

Lemma 3. *For every preorder $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$, we have*

$$x \setminus x \qquad \leftrightarrowtriangle \qquad x \qquad \leftrightarrowtriangle \qquad x \mathbin{/} x.$$

Proof. First, we show $x \leftrightarrowtriangle x \setminus x$ by proving both inclusions:

- $x \rightarrowtriangle x \setminus x \Leftrightarrow x \mathbin{;} x \rightarrowtriangle x$ which is transitivity;
- $x \setminus x \leftrightarrowtriangle 1 \mathbin{;} x \setminus x \rightarrowtriangle x \mathbin{;} x \setminus x \rightarrowtriangle x$ holds because $1 \rightarrowtriangle x$ is reflexivity.

So for every preorder $x \leftrightarrowtriangle x \setminus x$. Since the converse of a preorder is a preorder, it means that $x^{\smile} \leftrightarrowtriangle x^{\smile} \setminus x^{\smile}$. If we apply the converse to both sides we obtain $(x^{\smile})^{\smile} \leftrightarrowtriangle (x^{\smile} \setminus x^{\smile})^{\smile}$, therefore:

$$x \leftrightarrowtriangle (x^{\smile})^{\smile} \leftrightarrowtriangle (x^{\smile} \setminus x^{\smile})^{\smile} \leftrightarrowtriangle (x^{\smile})^{\smile} \mathbin{/} (x^{\smile})^{\smile} \leftrightarrowtriangle x \mathbin{/} x.$$ □

Preorders and reflexive transitive closures

Lemma 4. *For every $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$, the relation $x^{\star}$ is a preorder.*

Proof. Both implications:

- $1_{\mathrm{A}} \rightarrowtriangle 1_{\mathrm{A}} + x \mathbin{;} x^{\star} \rightarrowtriangle x^{\star}$
- $x \mathbin{;} x^{\star} \rightarrowtriangle 1_{\mathrm{A}} + x \mathbin{;} x^{\star} \rightarrowtriangle x^{\star}$ therefore $x^{\star} \mathbin{;} x^{\star} \rightarrowtriangle x^{\star}$. □

Lemma 5. *A relation $x : \mathrm{A} \mathrel{-\!\!\ominus\!\!\rightarrow} \mathrm{A}$ is a preorder iff $x \leftrightarrowtriangle x^{\star}$.*

Proof. First, observe that $x \rightarrowtriangle x^{\star}$ always holds. We then proceed as usual:

- If x is a preorder, then $x \mathbin{;} x \rightarrowtriangle x$, meaning $x \mathbin{;} x^{\star} \rightarrowtriangle x$. As such we derive $x^{\star} \leftrightarrowtriangle 1 \mathbin{;} x^{\star} \rightarrowtriangle x \mathbin{;} x^{\star} \rightarrowtriangle x$.
- Assume $x \leftrightarrowtriangle x^{\star}$. Then $x^{\star}$ being a preorder means x is a preorder. □

Composition of preorders

Lemma 6. *Given a pair of preorders* x, y *over the same set* A, $x \,;\, y$ *is a preorder iff* $y \,;\, x \rightarrowtriangle x \,;\, y$.

Proof. First, notice that $x \,;\, y$ is always reflexive, given that both x and y are:

$$1_{\mathrm{A}} \leftrightarrowtriangle 1_{\mathrm{A}} \,;\, 1_{\mathrm{A}} \rightarrowtriangle x \,;\, y.$$

We now (hopefully) prove both implications.

- Right to left: since $x \,;\, y$ is reflexive, we only need to check transitivity:

$$x \,;\, y \,;\, x \,;\, y \rightarrowtriangle x \,;\, x \,;\, y \,;\, y \rightarrowtriangle x \,;\, y.$$

- Left to right: since x and y are reflexive and $x \,;\, y$ is transitive:

$$y \,;\, x \leftrightarrowtriangle 1_{\mathrm{A}} \,;\, y \,;\, x \,;\, 1_{\mathrm{A}} \rightarrowtriangle x \,;\, y \,;\, x \,;\, y \rightarrowtriangle x \,;\, y.$$

□

Lemma 7. *Given a pair of preorders* $x, y : \mathrm{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathrm{A}$, $x \,;\, y$ *is a preorder iff* $x \,;\, y \leftrightarrowtriangle (x + y)^{\star}$.

Proof. We prove both implications.

- Assume $x \,;\, y$ is a preorder.
  - $x \,;\, y \rightarrowtriangle (x + y) \,;\, (x + y) \rightarrowtriangle (x + y)^{\star}$.
  - First, we observe that $x + y \leftrightarrowtriangle (x \,;\, 1) + (1 \,;\, y) \rightarrowtriangle (x \,;\, y) \,;\, (x \,;\, y)$, so by transitivity $x + y \rightarrowtriangle x \,;\, y$. By monotonicity of $-^{\star}$ we obtain $(x + y)^{\star} \rightarrowtriangle (x \,;\, y)^{\star}$.

    We now prove that $(x \,;\, y)^{\star} \rightarrowtriangle x \,;\, y$. This is entailed by $(x \,;\, y) \,;\, (x \,;\, y) \rightarrowtriangle x \,;\, y$, which is transitivity.
- Assume $x \,;\, y \leftrightarrowtriangle (x + y)^{\star}$. Since for any z, $z^{\star}$ is a preorder, in particular $(x + y)^{\star} \leftrightarrowtriangle x \,;\, y$ is a preorder. □