

# Representations

Séminaire 68nqrt

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# Disclaimer

- ☞ Work in progress
- ☞ Unpublished
- ☞ Largely a modelling/formalisation effort → many definitions, few results

## Context: Program analysis

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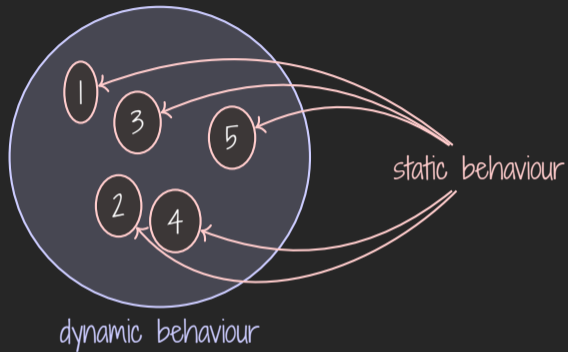
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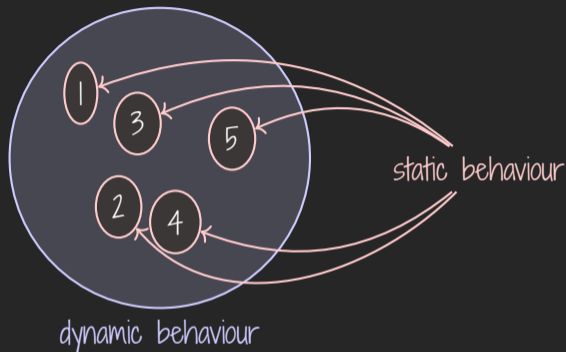
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- ☞ This work: machinery to build new frameworks compositionally.

# Two tiered models



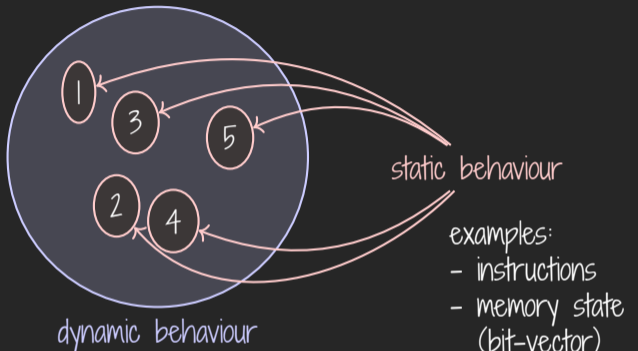
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examples:

- traces (i.e. words)
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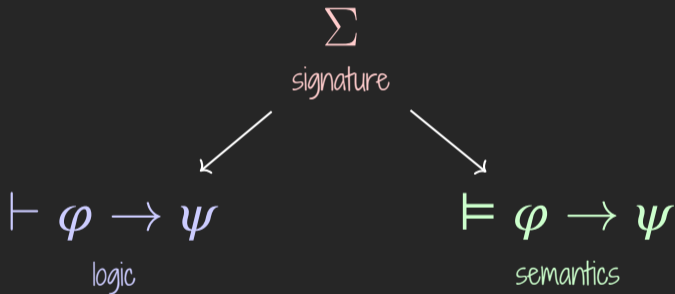
examples:

- instructions
- memory state (bit-vector)
- predicate/value record

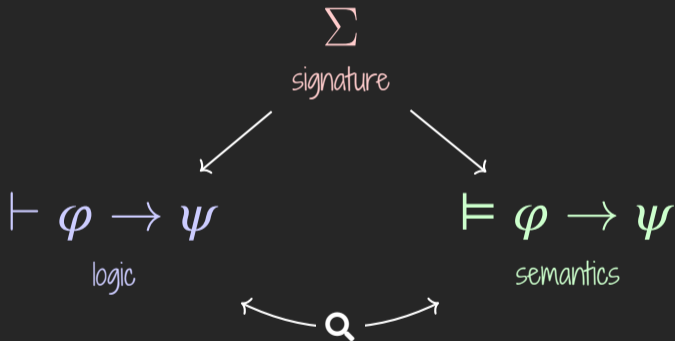
# Program analysis framework

$\Sigma$   
signature

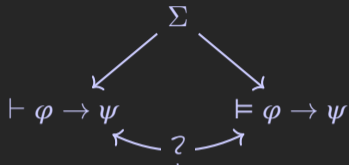
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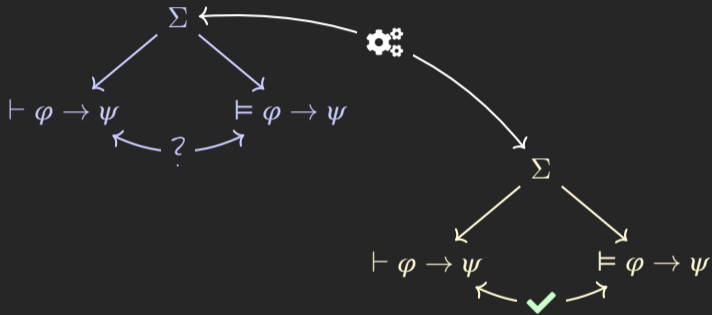
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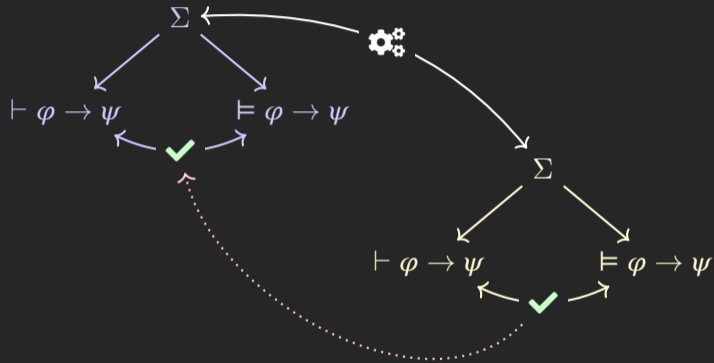
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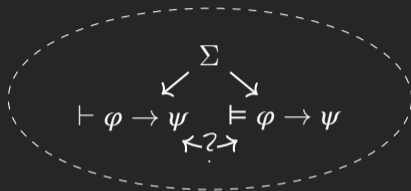
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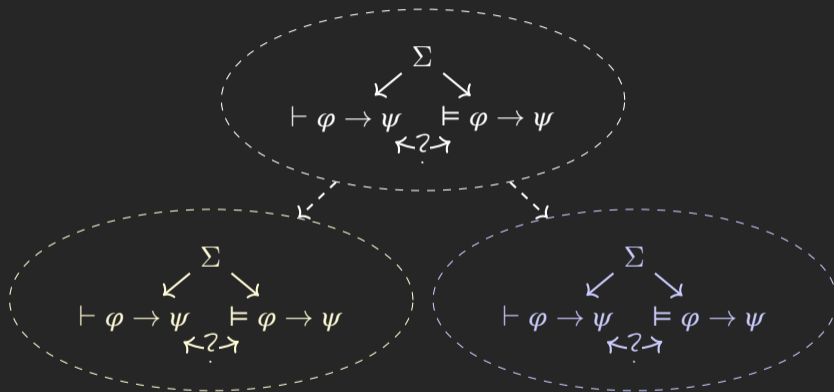
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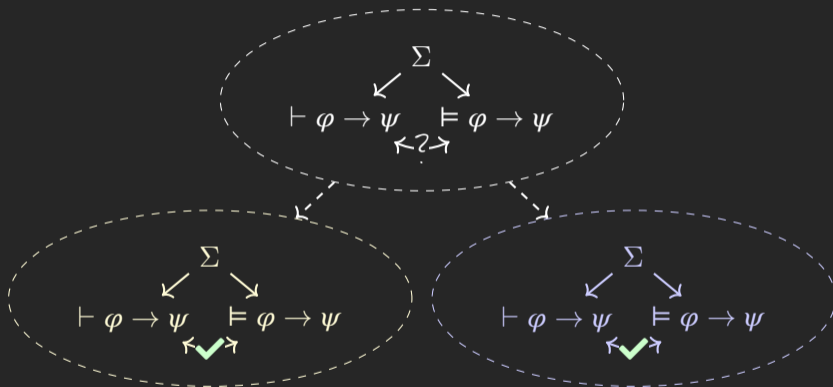
# Decompositions



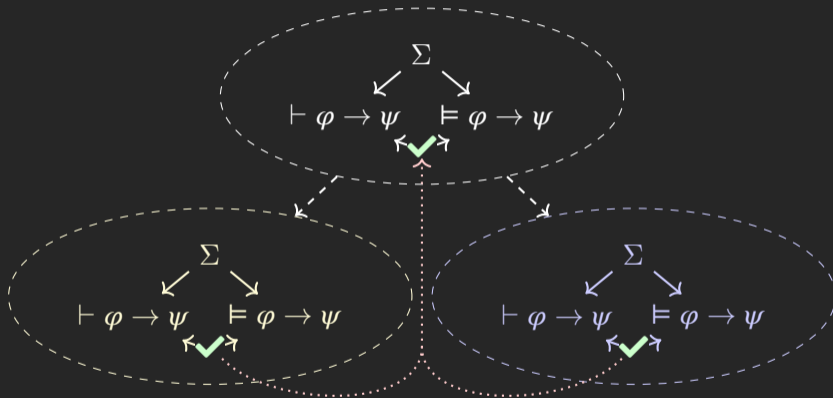
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# A meta-framework

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☞ Keep it simple:

- ▶ as few hypotheses as possible;
- ▶ diagrammatic or (in)equationnal reasoning.

# Representations

## Plan



I. Background

II. Representations

III. Reductions

IV. Higher order representations

V. Conclusion

# Background

- ☞ Category theory (a little)
- ☞ Relation algebra (a lot)

# Binary relations

$$x : A \rightarrow B \Leftrightarrow x \subseteq A \times B$$

i.e.  $x \subseteq y$

👉 entailment:  $x \rightarrow y$  means  $(a,b) \in x \Rightarrow (a,b) \in y$ .

👉 equivalence:  $x \leftrightarrow y$  means  $(a,b) \in x \Leftrightarrow (a,b) \in y$ .

i.e.  $x=y$

# Relation algebra

A multiplicative fragment

identity  $I_A : A \rightarrow A := \{(a, a') \mid a = a' \in A\}$ .

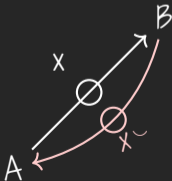


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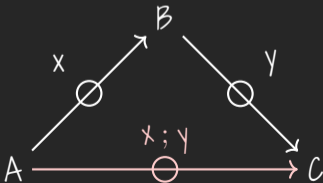
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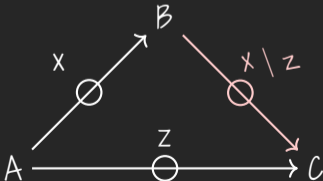
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left-residual  $x \setminus z : B \rightarrow C := \{(b, c) \mid \forall a \in A, (a, b) \in x \Rightarrow (a, c) \in z\}$ .

$$x ; y \rightarrow z \iff y \rightarrow x \setminus z$$



# Functions and relations

☞ Functions can be represented as relations:  
Let  $f : A \rightarrow B$  be a function

$$f_* : A \rightarrow B := \{(x, f(x)) \mid x \in A\}$$

$$\text{i.e. } (a, b) \in f_* \Leftrightarrow f(a) = b.$$

$$f^* : B \rightarrow A := \{(f(x), x) \mid x \in A\}$$

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☞ This is a monoid homomorphism:

$$(f \circ g)^* \Leftrightarrow f^* ; g^* \quad (\text{id}_A)^* \Leftrightarrow I_A$$

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☞ If a relation  $x$  is univalent and total there is a unique function  $f$  such that  $x = f_*$ .

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# Representations

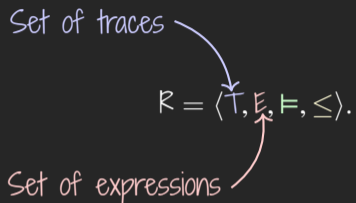
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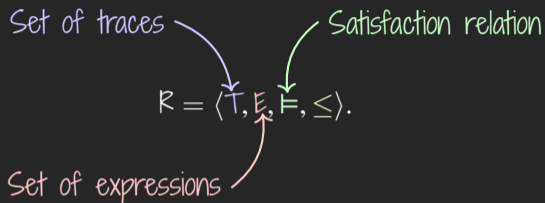
Set of traces


$$\mathcal{R} = \langle \mathcal{T}, E, F, \leq \rangle.$$

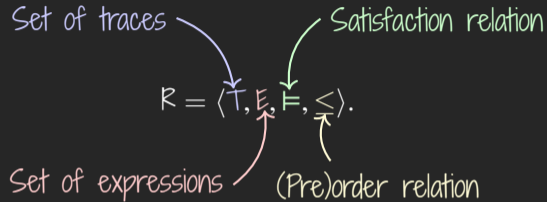
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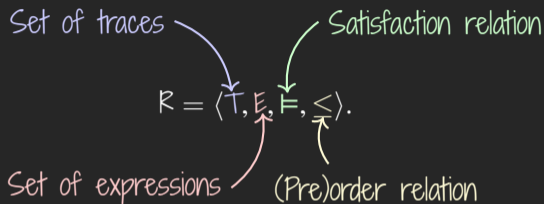
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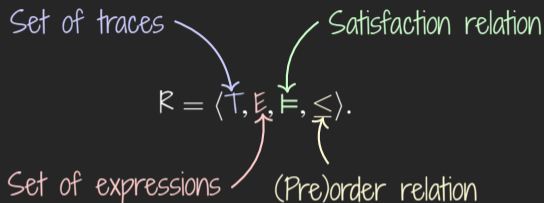


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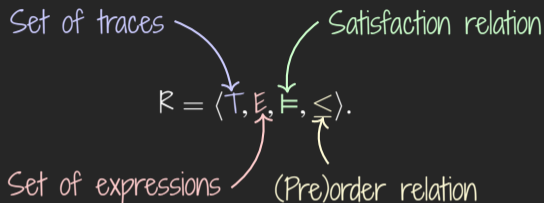
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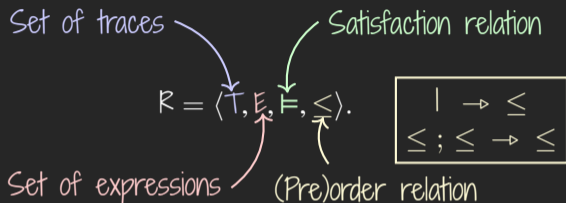


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$$\mathcal{R} = \langle \mathcal{T}, E, \models, \leq \rangle.$$

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To get back  $\models$ , we use the fact that  $t \models e \Leftrightarrow t \in I(e)$  i.e.  $\models := \in ; I^*$ .

# Interpretations

$$\mathcal{R} = \langle \mathcal{T}, \mathcal{E}, \vDash, \leq \rangle.$$

Equivalently, we can replace  $\vDash$  with an interpretation function  $l: \mathcal{E} \rightarrow \mathcal{P}(\mathcal{T})$ :

$$l(e) := \{t \in \mathcal{T} \mid t \vDash e\}$$

To get back  $\vDash$ , we use the fact that  $t \vDash e \Leftrightarrow t \in l(e)$  i.e.  $\vDash := \in ; l^*$ .

The axiom of representations means that  $\leq$  is sound with respect to  $l$ :

$$(\vDash ; \leq \rightarrow \vDash) \Leftrightarrow (\forall e, f \in \mathcal{E}, e \leq f \Rightarrow l(e) \subseteq l(f))$$

# Exact representations

A representation is exact if the following entailment holds:

$$\vDash \setminus \vDash \rightarrow \leq .$$

i.e.  $\forall e, f, (\forall t, t \vDash e \Rightarrow t \vDash f) \Rightarrow e \leq f$ .

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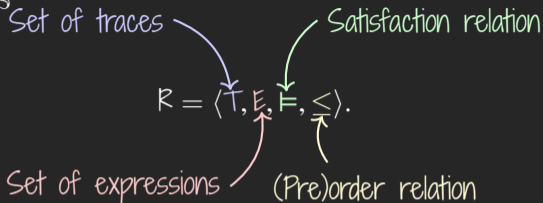
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## lemma

A representation is exact if and only if  $\leq$  is sound and complete for  $\vDash$ , i.e.:

$$\forall e, f \in E, \vDash(e) \subseteq \vDash(f) \Leftrightarrow e \leq f.$$

# Representations



$$| \rightarrow \leq$$

$$\leq ; \leq \rightarrow \leq$$

$$F ; \leq \rightarrow F$$

exact:

$$F \setminus F \rightarrow \leq$$

# Examples

$$\mathbb{R} = \langle T, E, \models, \leq \rangle.$$

- ☞ Algebra + axiomatisation
- ☞ Logic + semantics
- ☞ Programs + behavioural equivalence
- ☞ Programs + evaluation (exactness less relevant)

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Fact: for every binary relation  $x$ ,  $x \setminus x$  is a preorder.

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Example:


Since  $\in \setminus \in \leftrightarrow \subseteq$  (easy exercise)  $R^\in = \langle A, \mathcal{P}(A), \in, \subseteq \rangle$  is exact.

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A proof technique

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- ☞ To do that, I want to leverage the fact that I know some exact representations.
- ☞ I want to transfer the problem of checking exactness in  $R$  to the problem of proving exactness in  $R'$ .
- ☞ I need some transformations between  $R$  and  $R'$ .

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A proof technique

$$R_1 = \langle T_1, E_1, F_1, \leq_1 \rangle$$

$$R_2 = \langle T_2, E_2, F_2, \leq_2 \rangle$$

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A proof technique

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A reduction from  $R_1$  to  $R_2$  is given by:

✎ a pair of function  $\varphi : E_1 \rightarrow E_2$  and  $\psi : E_2 \rightarrow E_1$

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$$\boxed{\psi^*; \leq_2 \rightarrow \leq_1; \psi^* \quad \psi^*; \varphi^* \rightarrow \equiv_1 \quad F_2; \varphi^* \leftrightarrow \rho; F_1}$$

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$$\text{i.e. :} \quad e \leq_2 e' \Rightarrow \psi(e) \leq_1 \psi(e') \quad e \leq_1 \psi(\varphi(e)) \leq_1 e$$

$$h_2(\varphi(e)) = \{t \in T_2 \mid \exists s, (t, s) \in \rho \ \& \ s \in h_1(e)\}$$

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A representation  $R$  is exact iff  $R$  reduces to  $\langle T, E, F, F \setminus F \rangle$ .

## Proof

If  $R_2$  is exact and  $R_1$  reduces to  $R_2$ , then  $R_1$  is exact.

1) Show that  $\mathbb{F}_1 \setminus \mathbb{F}_1 \rightarrow \varphi_*; \mathbb{F}_2 \setminus \mathbb{F}_2; \varphi^*$ .

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$$\mathbb{F}_2; \varphi^*; (\mathbb{F}_1 \setminus \mathbb{F}_1) \leftrightarrow \rho; \mathbb{F}_1; (\mathbb{F}_1 \setminus \mathbb{F}_1) \rightarrow \rho; \mathbb{F}_1 \leftrightarrow \mathbb{F}_2; \varphi^*$$

2) Therefore, we have :

$$\begin{aligned} \mathbb{F}_1 \setminus \mathbb{F}_1 \rightarrow \varphi_*; \mathbb{F}_2 \setminus \mathbb{F}_2; \varphi^* &\rightarrow \varphi_*; \leq_2; \varphi^* \\ &\rightarrow \varphi_*; \psi_*; \psi^* \leq_2; \varphi^* \\ &\rightarrow \varphi_*; \psi_* \leq_1; \psi^*; \varphi^* \\ &\rightarrow \equiv_1; \leq_1; \equiv_1 \rightarrow \leq_1 \end{aligned}$$

□

## Reduction: example

We are interested in the following representation:

$$\text{FinLat}(A) = \langle A, E(A), \in ; !^*, \leq \rangle$$

$$e, f \in E(A) ::= a \mid \perp \mid e \vee f$$

$$!(a) := \{a\}$$

$$!(\perp) = \emptyset$$

$$!(e \vee f) = !(e) \cup !(f)$$

$$e \leq e$$

$$\emptyset \leq e$$

$$e \leq e \vee f$$

$$f \leq e \vee f$$

$$\frac{e \leq f \leq g}{e \leq g}$$

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Its soundness is obvious, we now want to prove it exact.

## Reduction: example

We start with a trivial representation

$$\text{FinSet}(A) = \langle A, A^*, \text{In}, \text{In} \setminus \text{In} \rangle$$

with  $\text{In}$  defined by  $(a, b \cdot w) \in \text{In} \Leftrightarrow a = b \vee (a, w) \in \text{In}$ .

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$\text{FinLat}$  reduces to  $\text{FinSet}$ .

$$\varphi(a) = a$$

$$\varphi(\perp) = \varepsilon$$

$$\varphi(e \vee f) = \varphi(e) \cdot \varphi(f)$$

$$\psi(\varepsilon) = \perp$$

$$\psi(a \cdot w) = a \vee \psi(w)$$

$$(\rho \leftrightarrow \downarrow_A)$$

FinLat reduces to FinSet

1)  $\psi^*; \leq_2 \rightarrow \leq_1; \psi^*$

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by induction on  $u$ :

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$$\Rightarrow a \cdot u \text{ (In } \setminus \text{ In) } v \Rightarrow a \text{ In } v \text{ and } u \text{ (In } \setminus \text{ In) } v$$

by induction on  $v$   $a \text{ In } v \Rightarrow a \leq \psi(v)$

by induction hypothesis  $\psi(u) \leq \psi(v)$

$$\Rightarrow \psi(a \cdot u) = a \vee \psi(u) \leq \psi(v)$$

FinLat reduces to FinSet

2)  $\psi^* ; \varphi^* \rightarrow \equiv_1$

FinLat reduces to FinSet

$$2) \psi^* ; \varphi^* \rightarrow \equiv_1$$

by induction on terms

$$\psi(\varphi(a)) = a \vee \perp \equiv a$$

$$\psi(\varphi(\perp)) = \psi(\varepsilon) = \perp$$

by induction on  $u$

$$\frac{}{\psi(u \cdot v) \equiv \psi(u) \vee \psi(v)}$$

$$\frac{}{\psi(\varphi(e \vee f)) = \psi(\varphi(e) \cdot \varphi(f)) \equiv \psi(\varphi(e)) \vee \psi(\varphi(f))}$$

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$$a \ln \varphi(b) = b \cdot \varepsilon \Leftrightarrow a = b \Leftrightarrow a \in \{b\} = l(b)$$

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$$\neg(a \in \emptyset = l(\perp))$$

$$\begin{aligned} a \ln \varphi(e \vee f) = \varphi(e) \cdot \varphi(f) &\Leftrightarrow a \ln \varphi(e) \text{ or } a \ln \varphi(f) \\ &\Leftrightarrow a \in l(e) \text{ or } a \in l(f) \\ &\Leftrightarrow a \in l(e) \cup l(f) = l(e \vee f) \end{aligned}$$

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# Representations

## Plan

I. Background

II. Representations

III. Reductions

 IV. Higher order representations

V. Conclusion

# Higher order representations

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- ☞ e.g.  $\text{FinLat}(A)$ ,  $\text{FinSet}(A)$ ,  $\text{Lang}(A)$ ,  $\text{Mon}(A)$

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# The category Repr

Representation morphisms

$$\langle T_1, E_1, F_1, \leq_1 \rangle \xrightarrow{\langle f, g \rangle} \langle T_2, E_2, F_2, \leq_2 \rangle.$$

with  $f : T_1 \rightarrow T_2$  and  $g : E_1 \rightarrow E_2$ , and such that:

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$$\forall e, f \in E_1, e \leq_1 f \Rightarrow g(e) \leq_2 g(f).$$

$$f^*; F_1 \leftrightarrow F_2 ; g^*$$

$$\forall e \in E_1, \{f(t) \mid t F_1 e\} = \{t \in T_2 \mid t F_2 g(e)\}, \text{ i.e. } \iota_2(g(e)) = \{f(t) \mid t \in \iota_1(e)\}$$

# The category $\text{Repr}$

☞ Objects: representations

☞ Arrows: morphisms

Remark: exact representations form a full sub-category of  $\text{Repr}$ .

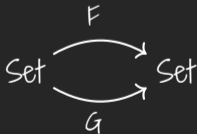
# Higher order representations

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 $\rightarrow$  "higher order relations"
- ☞ Parametric representations over representations
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# Higher order relations

Let  $F, G$  be Set endofunctors.

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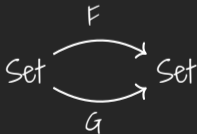
☞ Set-indexed family of relations  $\rho_A : FA \rightrightarrows GA$ .

☞ notation :  $\rho : F \rightrightarrows G$  if for all  $f : A \rightarrow B$

"natural relation"

$$(x, y) \in \rho_A \Rightarrow (Ff(x), Gf(y)) \in \rho_B$$

$$\text{i.e. } \rho ; Gf_* \rightarrow Ff_* ; \rho$$



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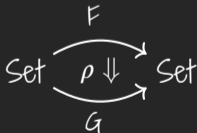
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☞  $\approx$  2-cell in Rel

☞ but not quite:  $F, G$  need to be  $\text{Set} \rightarrow \text{Rel}$  functors!



☞  $F_* : \text{Set} \rightarrow \text{Rel}$  does as  $F$  on objects, and maps  $f$  to  $(Ff)_*$ .

# Relation lifting

Let  $F : \text{Set} \rightarrow \text{Set}$  be a functor preserving weak pullbacks.  
 $F$  can be lifted to a functor  $\bar{F} : \text{Rel} \rightarrow \text{Rel}$  such that:

$$\bar{F}(f^*) \leftrightarrow (Ff)^*$$

$$\frac{x \rightarrow y}{\bar{F}x \rightarrow \bar{F}y}$$

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lemma

$F$  extends as a monotone functor  $\text{Rel} \rightarrow \text{Rel}$  iff it preserves weak pullbacks.

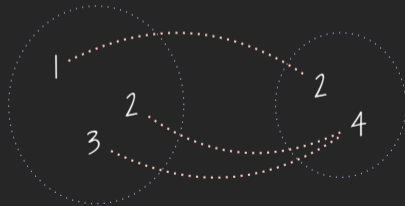
# Example: $\bar{\mathcal{P}}(x)$

$$(A, B) \in \bar{\mathcal{P}}(x) \Leftrightarrow \begin{cases} \forall a \in A, \exists b \in B : (a, b) \in x \\ \forall b \in B, \exists a \in A : (a, b) \in x. \end{cases}$$

$\{1, 2, 3\} \bar{\mathcal{P}}(<) \{2, 4\}$ .

$\{0\} \bar{\mathcal{P}}(<) \{2, 4\}$ .

$(\{1, 2, 3\}, \{0, 4\}) \notin \bar{\mathcal{P}}(<)$ .



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## Example: $\bar{T}_\Sigma(X)$

Let  $\Sigma$  be some signature with finite arities.

$$e \in T_\Sigma(A) ::= \text{var}(a) \mid f(e_1, \dots, e_n) \quad \text{with } a \in A, f^{(n)} \in \Sigma.$$

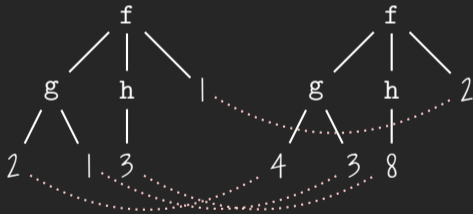
$$\frac{(a, b) \in X}{(\text{var}(a), \text{var}(b)) \in \bar{T}_\Sigma(X)}$$

$$\frac{\forall i (e_i, f_i) \in \bar{T}_\Sigma(X)}{(f(e_1, \dots, e_n), f(f_1, \dots, f_n)) \in \bar{T}_\Sigma(X)}$$

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Example:  $\Sigma = \{f^{(3)}, g^{(2)}, h^{(1)}\}$  and  $A = \mathbb{N}$ .

$$f(g(2,1), h(3), l) \bar{T}_\Sigma(\langle) f(g(4,3), h(8), 2)$$

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$$\rho ; Gf_* \rightarrow Ff_* ; \rho$$

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$$\rho ; \bar{G}x \leftrightarrow \bar{F}x ; \rho$$

# Higher order representations

A HO-representation is given by:

☞ a pair of Set-endofunctors  $\mathcal{E}$  and  $\mathcal{T}$ ;

☞ a right-linear relation  $\Xi: \mathcal{T} \rightleftarrows \mathcal{E}$ ;

☞ a natural relation  $\leq: \mathcal{E} \rightleftarrows \mathcal{E}$ ;

$$\forall x, \Xi; \bar{\mathcal{E}}x \rightarrow \bar{\mathcal{T}}x; \Xi$$

$$\forall f, \mathcal{E}f^*; \leq \rightarrow \leq; \mathcal{E}f^*$$

such that for every set  $A$ ,  $\mathcal{R}(A) := \langle \mathcal{E}(A), \mathcal{T}(A), \Xi_A, \leq_A \rangle$  is a representation.

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Equivalent to the interpretation function being a natural transformation  $\mathcal{E} \Rightarrow \mathcal{P}\mathcal{T}$ .

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## Theorem

HO-representations are exactly the functors  $\text{Set} \rightarrow \text{Repr}$ .

# Overlaying representations

Can we use HO-representations to get functors  $\text{Repr} \rightarrow \text{Repr}$ ?

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Yes! Let's do it!

# Overlaid representations

Let  $\mathcal{R} = \langle \mathcal{T}, \mathcal{E}, \mathbb{E}, \leq \rangle$  be an HO-representation.

$\widehat{\mathcal{R}}$  is defined on an object  $R = \langle \mathcal{T}, \mathcal{E}, \mathbb{E}, \leq \rangle$  as

$$\widehat{\mathcal{R}}(R) := \langle \mathcal{T}(\mathcal{T}), \mathcal{E}(\mathcal{E}), \mathbb{E}, \lesssim \rangle.$$

with:  $\mathbb{E} := \mathcal{T}(\mathbb{E})$ ;  $\mathbb{E}$  and  $\lesssim := (\mathcal{E} \leq + \leq)^\star$ .  
reflexive-transitive closure

The image of a morphism is  $\widehat{\mathcal{R}}\langle f, g \rangle := \langle \mathcal{T}f, \mathcal{E}g \rangle$ .

# Overlaid representations

Let  $\mathcal{R} = \langle \mathcal{T}, \mathcal{E}, \models, \leq \rangle$  be an HO-representation.

$\widehat{\mathcal{R}}$  is defined on an object  $R = \langle T, E, F, \leq \rangle$  as

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lemma

$\widehat{\mathcal{R}}$  is a functor  $\text{Repr} \rightarrow \text{Repr}$ .

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- ✎ We can get exactness preserving functors on sub-categories, e.g. ideal representations:

$$\langle T, E, \sqsubseteq, \iota, \leq \rangle$$

with  $\sqsubseteq$  a preorder on  $T$  and  $\iota : E \rightarrow T$ .

$$\models \leftrightarrow \sqsubseteq ; \iota^*$$

i.e.  $\models(e) = \{t \in T \mid t \sqsubseteq \iota(e)\}$ .

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i.e.  $\models(e) = \{t \in T \mid t \trianglelefteq \iota(e)\}$ .

- ☞ More work to be done...

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- ☞ Bundling together syntax + semantics
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- ☞ Within reach: multi Kleene algebra, with  $n$  sequential products/stars and  $m$  parallel products/stars.

That's all folks!

Merci de votre attention!

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