Representations: a general theory

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Representations: a general theory

Slogan

Relation algebra is a good meta-langage for system analysis.

Source:

Representations

A meta-model for system analysis

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Abstract. The formal analysis of automated systems is an important and growing industry. This activity routinely requires new verification frameworks to be developed to tackle new programming features, or

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Important example:

- $\in \setminus \in \;\leftrightarrow\; \subseteq$

Preorders

> <u>What we learned in school</u>
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> A preorder over a set A is a <u>reflexive</u> and <u>transitive</u> relation.

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<u>Alternatively</u>: $x : A \nrightarrow A$ such that $x \leftrightarrow x \backslash x$.

<u>Or</u>: any relation of the form $x \backslash x$, for $x : B \nrightarrow A$.

Specification formalism

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$[\![\varphi]\!]_{\models} := \{\mathfrak{m} \in \mathcal{M} \mid \mathfrak{m} \models \varphi\}$ .... interpretation function

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$\varphi \lesssim \psi$ means $[\![\varphi]\!] \subseteq [\![\psi]\!]$

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complete order $\longrightarrow$ <u>exact</u> representation

In “practice”

$\leqslant$ can be several things:

- result of some algorithm $\leqslant := \{(n, m) \mid \texttt{compare}(n, m) = \texttt{true}\}$;
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Exact representations $\rightarrow$ characterising semantic containment.

Examples

☞ FO logic:

- fix some (functional) signature Σ
- let $\mathcal{M}$ be relational structures over Σ
- let Φ be closed FO formulas over Σ
- $\models$ is defined as usual
- $\varphi \leqslant \psi$ means $\exists \pi : \varphi \vdash \psi$

Gödel's completeness theorem:

☞ This is an exact representation.

Examples

- Kleene algebra
  - fix some alphabet A
  - let $\mathcal{M} := A^\star$ be words over A
  - let $\Phi := \mathsf{Reg}(A)$ be the set of regular expressions over A
  - $w \vDash e$ means $w \in \mathcal{L}(e)$
  - $e \leqslant f$ means $\mathsf{KA} \vdash e \leqslant f$ where KA is e.g. Kozen's axioms

Kozen's completeness theorem:

- This is an exact representation.

Examples

☞ Automata comparison

- fix some finite alphabet A
- $\mathcal{M} := A^\star$
- take DFAs over A as formulae Φ
- $w \models \mathcal{A}$ iff $w \in [\![\mathcal{A}]\!]$
- $\mathcal{A} \leqslant \mathcal{B}$ iff $\mathcal{B}$ can simulate $\mathcal{A}$

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Many proofs from the litterature can be recast as instances:

- ☞ Proof of completeness of KAT [Kozen & Smith, CSL 1997]
- ☞ Proof of completeness of SKA [Wagemaker et al., MPC 2019]
- ☞ Proof of completeness of CKA [Kappé et al., CONCUR 2018]
- ☞ reductions in KA+hypothesis [Pous et al., RAMiCS 2021]

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- a Cartesian category of representations;
- parametric representations ≈ functors $\mathbf{Set} \to \mathbf{Repr}$
- lifting these to $\mathbf{Repr} \to \mathbf{Repr}$

$\to$ tiered representations

For the future

☞ a calculus of representations

For the future

- a calculus of representations
- a library of formal proofs

For the future

- a calculus of representations
- a library of formal proofs
- ...insert your research here...

Thanks

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