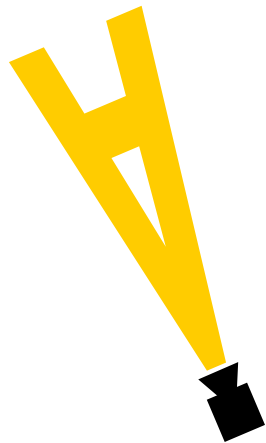


REPRESENTATIONS: A GENERAL THEORY

Highlights, September 2026

Paul Brunet

Université Paris-Est Créteil Val de Marne



REPRESENTATIONS: A GENERAL THEORY

1

Slogan

Relation algebra is a good meta-language for
system analysis.

Source:

Representations

A meta-model for system analysis

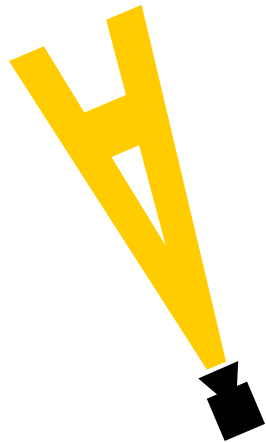
Paul Brunet^[0000-0002-9762-6872]

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<https://paul.brunet-zamansky.fr>

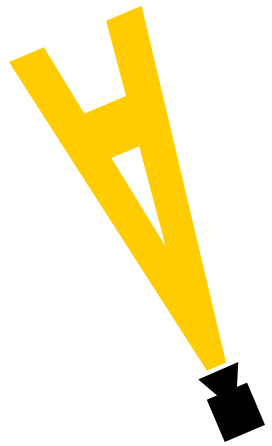
Abstract. The formal analysis of automated systems is an important and growing industry. This activity routinely requires new verification frameworks to be developed to tackle new programming features, or



RELATION ALGEBRA

Binary relations ~ sets of pairs:

👉 $x : A \rightarrow B$ means $x \subseteq A \times B$



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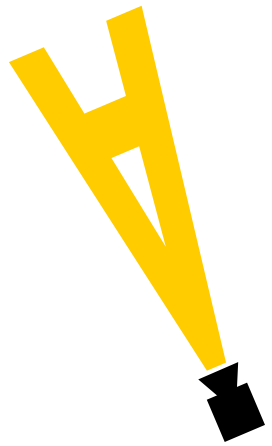
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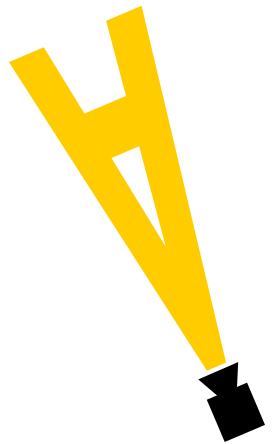
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Operators:

👉 identity $1_A : A \rightarrow A$ (a.k.a. diagonal)

$$1_A := \{(a, a) \mid a \in A\}$$



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$$\begin{cases} x:A \rightarrow B \\ y:B \rightarrow C \\ z:A \rightarrow C \end{cases}$$

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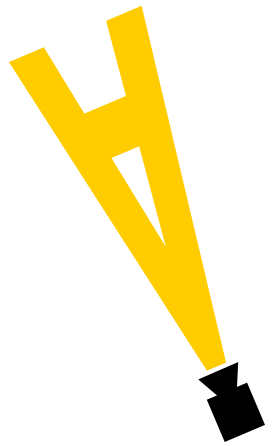
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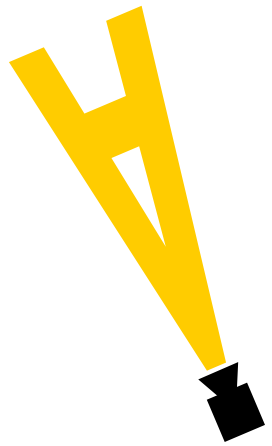
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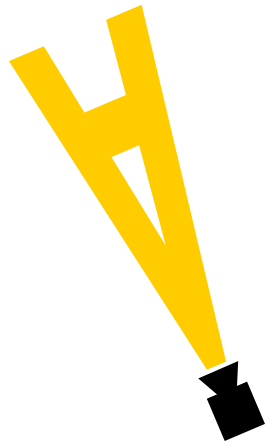
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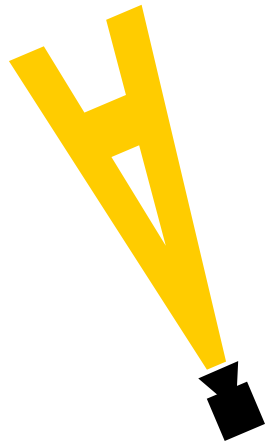
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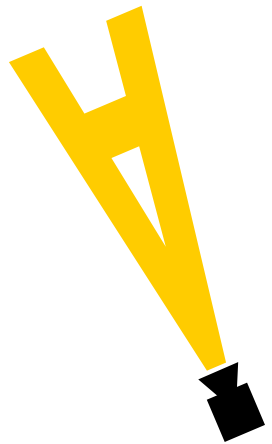
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Important example:

👉 $\in \setminus \in \leftrightarrow \subseteq$

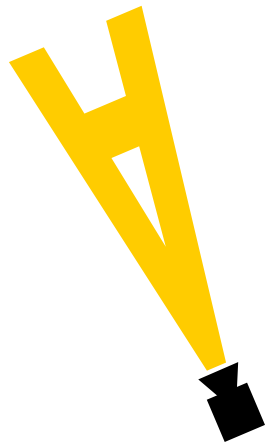


PREORDERS

What we learned in school

A preorder over a set A is a reflexive and transitive relation.

Relationally: $x : A \Rightarrow A$ such that $1_A \rightarrow x$ (ref.) and $x ; x \rightarrow x$ (trans.).



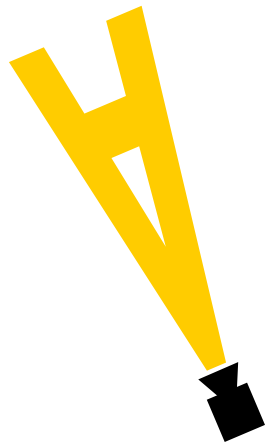
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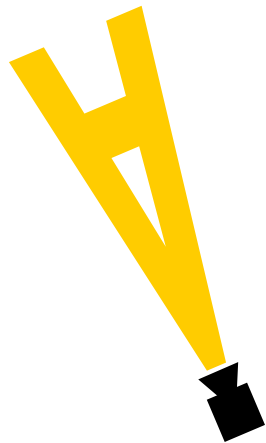
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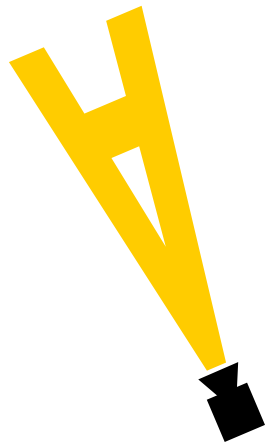
or: any relation of the form $x \mid x$, for $x : B \Rightarrow A$.



SPECIFICATION FORMALISM

4

$$\models: \mathcal{M} \rightarrow \Phi$$

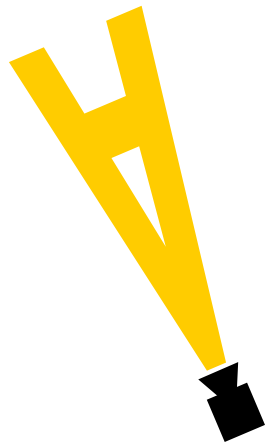


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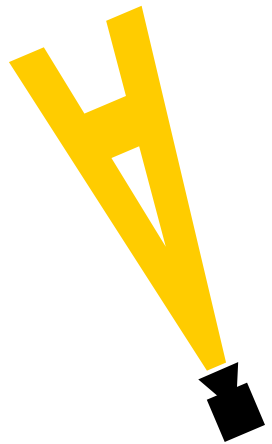
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Definitions

$\llbracket \varphi \rrbracket_{\models} := \{m \in \mathcal{M} \mid m \models \varphi\}$ interpretation function

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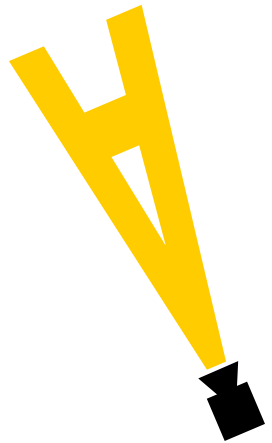
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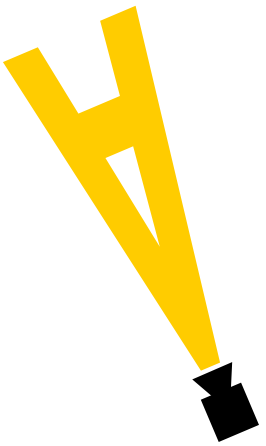
☞ $\varphi \lesssim \psi$ means $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$



REPRESENTATIONS

5

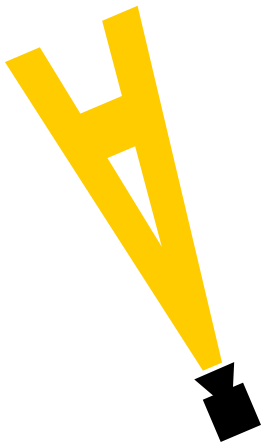
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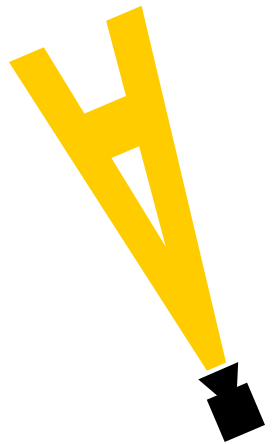
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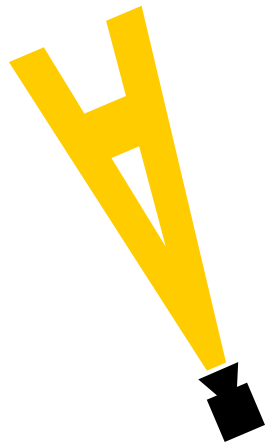
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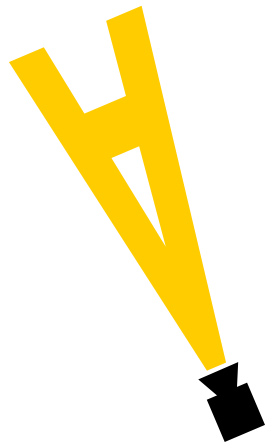
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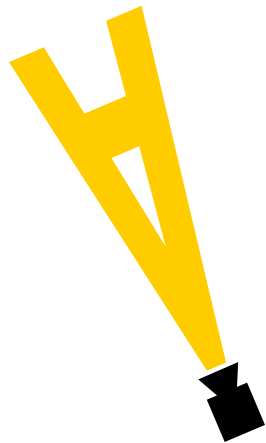
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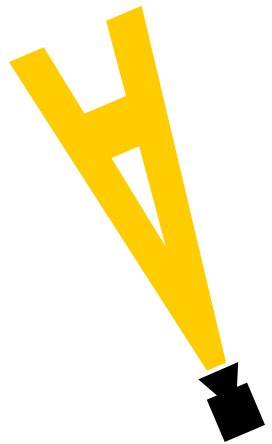
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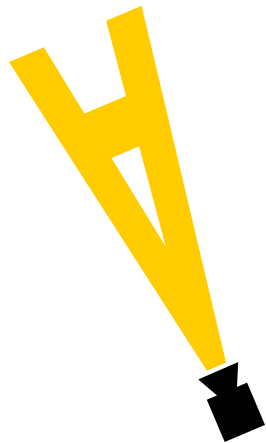
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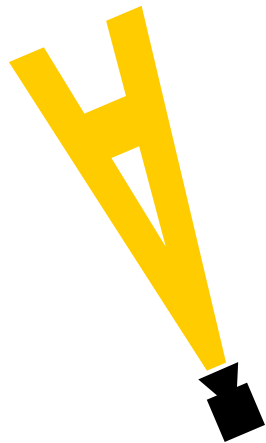
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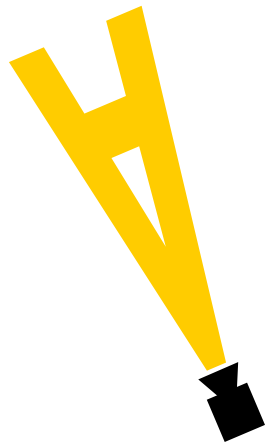


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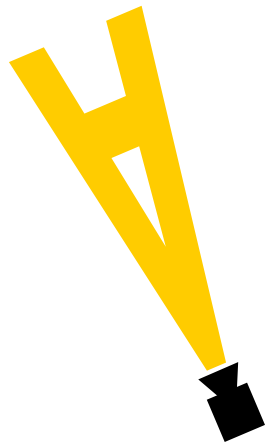
complete order $\rightarrow$ exact representation



IN "PRACTICE"

$\leq$ can be several things:

- 👉 result of some algorithm $\leq := \{(n, m) \mid \text{compare}(n, m) = \text{true}\};$
- 👉 axiomatic relation $\leq := \{(\varphi, \psi) \mid \vdash \varphi \rightarrow \psi\};$
- 👉 ...



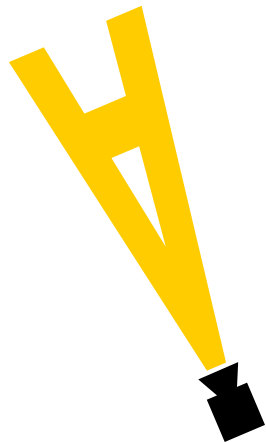
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Exact representations $\rightarrow$ characterising semantic containment.



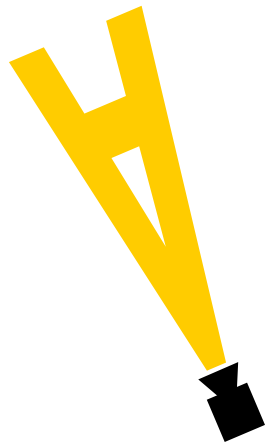
EXAMPLES

👉 FO logic:

- fix some (functional) signature Σ
- let $\mathcal{M}$ be relational structures over Σ
- let Φ be closed FO formulas over Σ
- $\models$ is defined as usual
- $\varphi \leq \psi$ means $\exists \pi : \varphi \vdash \psi$

Gödel's completeness theorem:

👉 This is an exact representation.

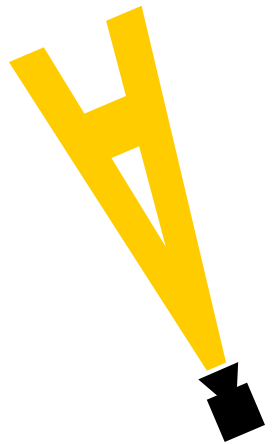


EXAMPLES

- ☞ Kleene algebra
- fix some alphabet A
 - let $\mathcal{M} := A^*$ be words over A
 - let $\Phi := \text{Reg}(A)$ be the set of regular expressions over A
 - $w \models e$ means $w \in \mathcal{L}(e)$
 - $e \leq f$ means $\text{KA} \vdash e \leq f$ where KA is e.g. Kozen's axioms

Kozen's completeness theorem:

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EXAMPLES



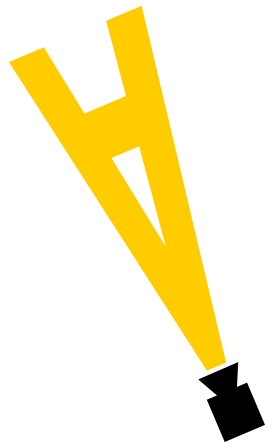
Automata comparison

- fix some finite alphabet A
- $\mathcal{M} := A^*$
- take DFAs over A as formulae Φ
- $w \models \mathcal{A}$ iff $w \in \llbracket \mathcal{A} \rrbracket$
- $\mathcal{A} \leq \mathcal{B}$ iff $\mathcal{B}$ can simulate $\mathcal{A}$

Rutten:



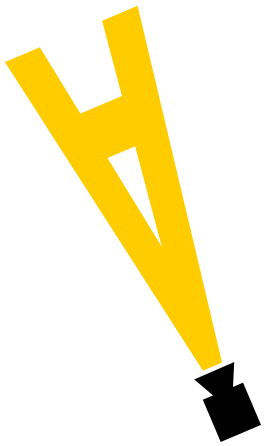
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REDUCTIONS

10

A deductively complete proof technique to establish exactness.



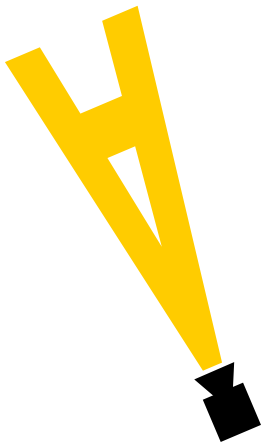
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Lemma



R is exact iff there is a reduction from R to some exact representation.



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A deductively complete proof technique to establish exactness.

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Many proofs from the literature can be recast as instances:

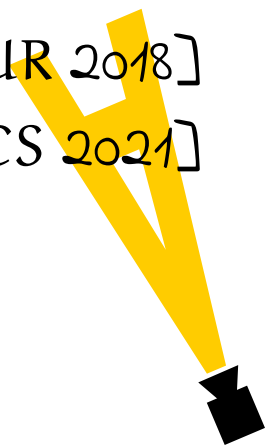
- 👉 Proof of completeness of KAT
- 👉 Proof of completeness of SKA
- 👉 Proof of completeness of CKA
- 👉 reductions in KA+hypothesis

[Kozen & Smith, CSL 1997]

[Wagemaker et al., MPC 2019]

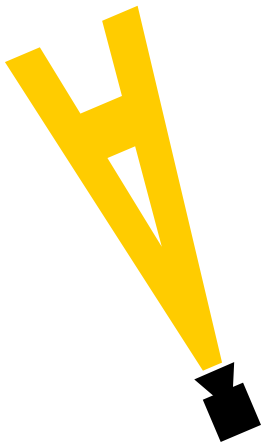
[Kappé et al., CONCUR 2018]

[Pous et al., RAMiCS 2021]



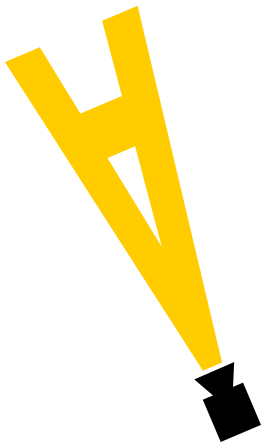
ALSO IN THE PAPER

👉 a Cartesian category of representations;



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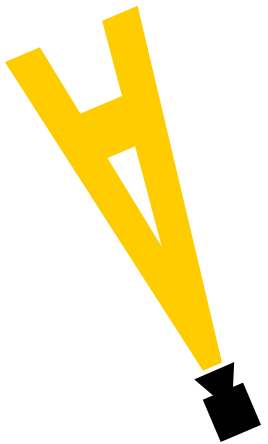
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- 👉 parametric representations $\approx$ functors $\text{Set} \rightarrow \text{Repr}$



ALSO IN THE PAPER

- 👉 a Cartesian category of representations;
- 👉 parametric representations $\approx$ functors $\text{Set} \rightarrow \text{Repr}$
- 👉 lifting these to $\text{Repr} \rightarrow \text{Repr}$

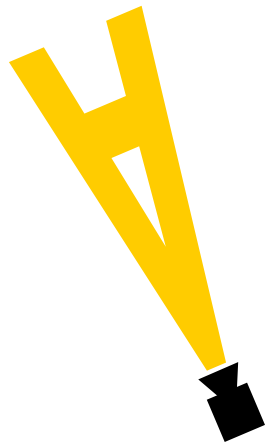
→ tiered representations



FOR THE FUTURE

12

👉 a calculus of representations

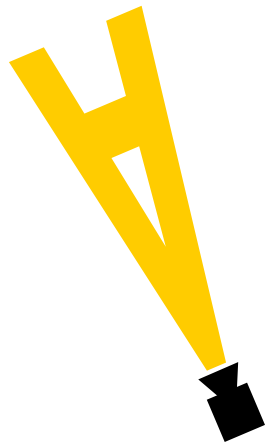


FOR THE FUTURE

12

👉 a calculus of representations

👉 a library of formal proofs



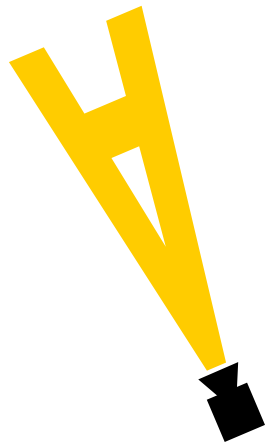
FOR THE FUTURE

12

👉 a calculus of representations

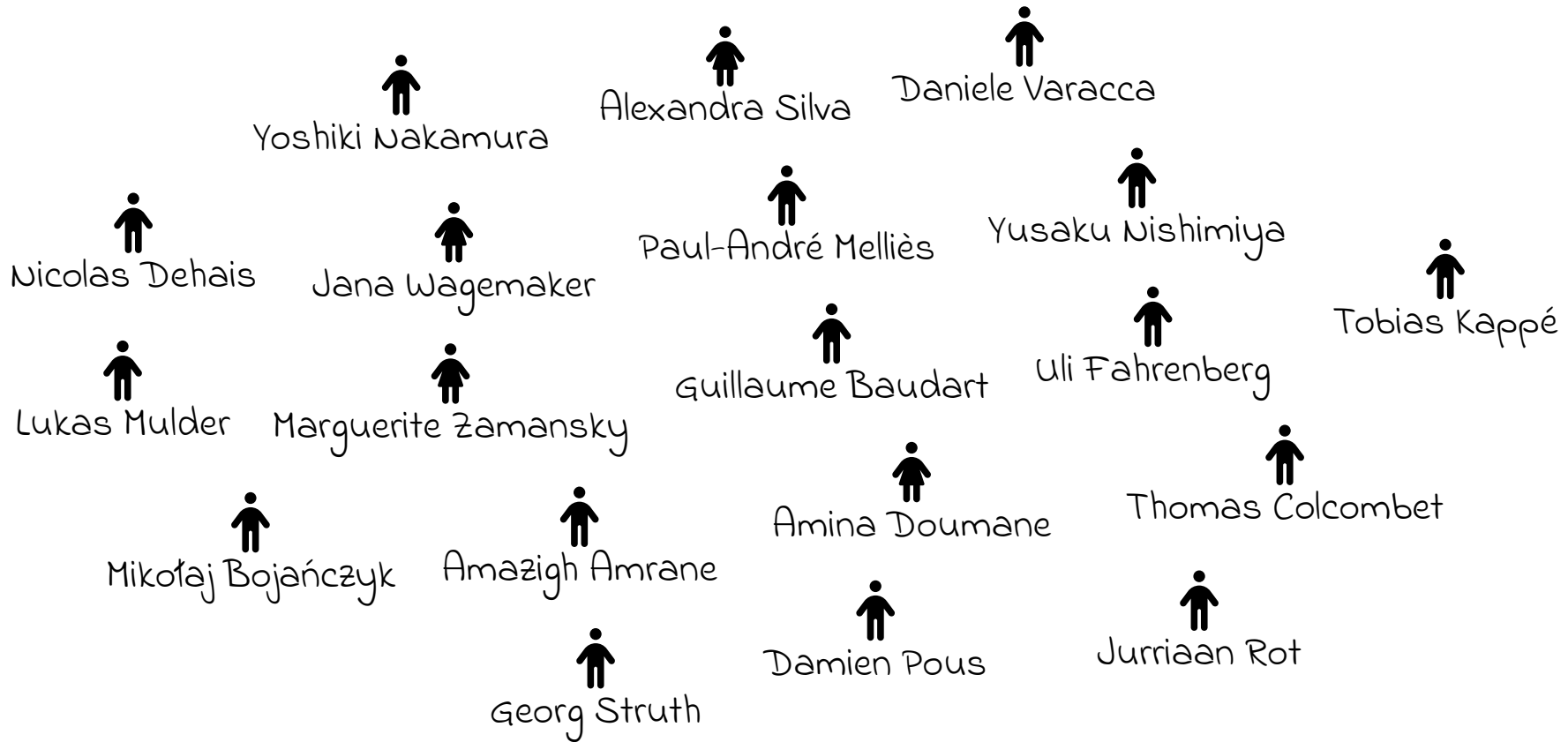
👉 a library of formal proofs

👉 ...insert your research here...



THANKS

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