A Kleene Theorem for Petri automata

Highlights
of Logic, Games and Automata

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Kleene Theorem

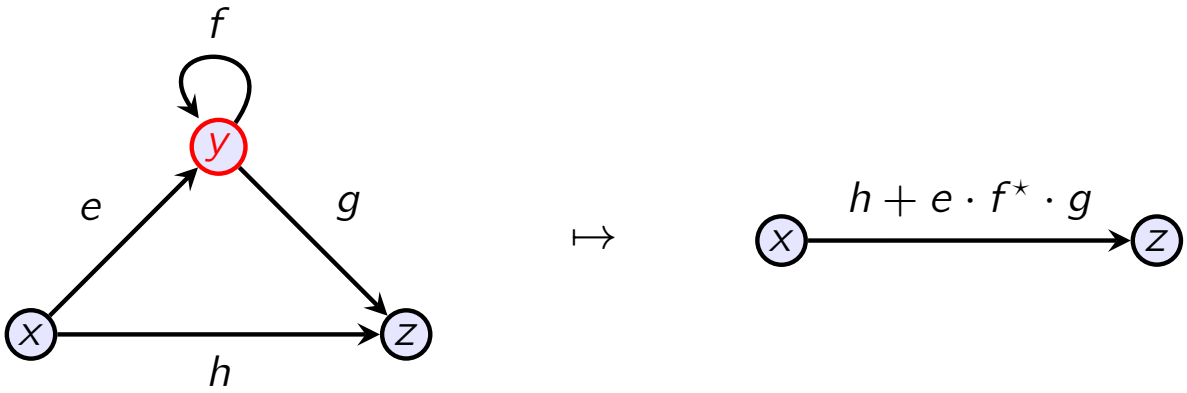
Theorem

Let $L \subseteq \Sigma^\star$ be a language. The following are equivalent:

- there is a **regular expression** e such that $L = [\![e]\!]$;
- there is a **finite state automaton** $\mathcal{A}$ such that $L = \mathcal{L}(\mathcal{A})$.

Proof.

- from expressions to automata: inductive construction
- from automata to expressions:
  - switch to **generalised automata** by changing the labels
  - eliminate states one after the other, while enriching transitions to keep the same overall language.



Outline

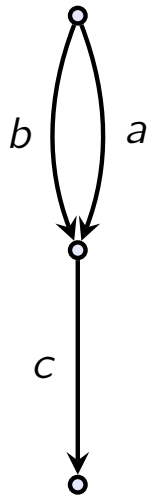
Outline

Series Parallel graphs and terms

Our "words"

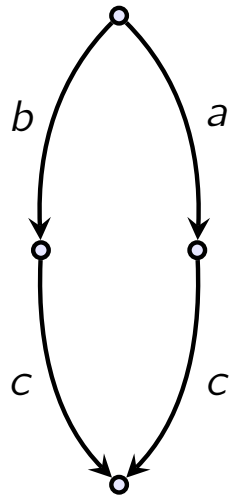
$$u, v ::= a \mid u \cdot v \mid u \cap v$$

$(a \cap b) \cdot c$



$G((a \cap b) \cdot c)$

$(a \cdot c) \cap (b \cdot c)$



$G((a \cdot c) \cap (b \cdot c))$

Regular graph languages

Regular lattice expressions

$$e, f ::= 0 \mid a \mid e + f \mid e \cdot f \mid e^{+} \mid e \cap f$$

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Graph language

$$\mathcal{G}(a) = \{G(a)\} \qquad \mathcal{G}(e \cdot f) = \{E \cdot F \mid \langle E, F \rangle \in \mathcal{G}(e) \times \mathcal{G}(f)\}$$

$$\mathcal{G}(e \cap f) = \{E \cap F \mid \langle E, F \rangle \in \mathcal{G}(e) \times \mathcal{G}(f)\}$$

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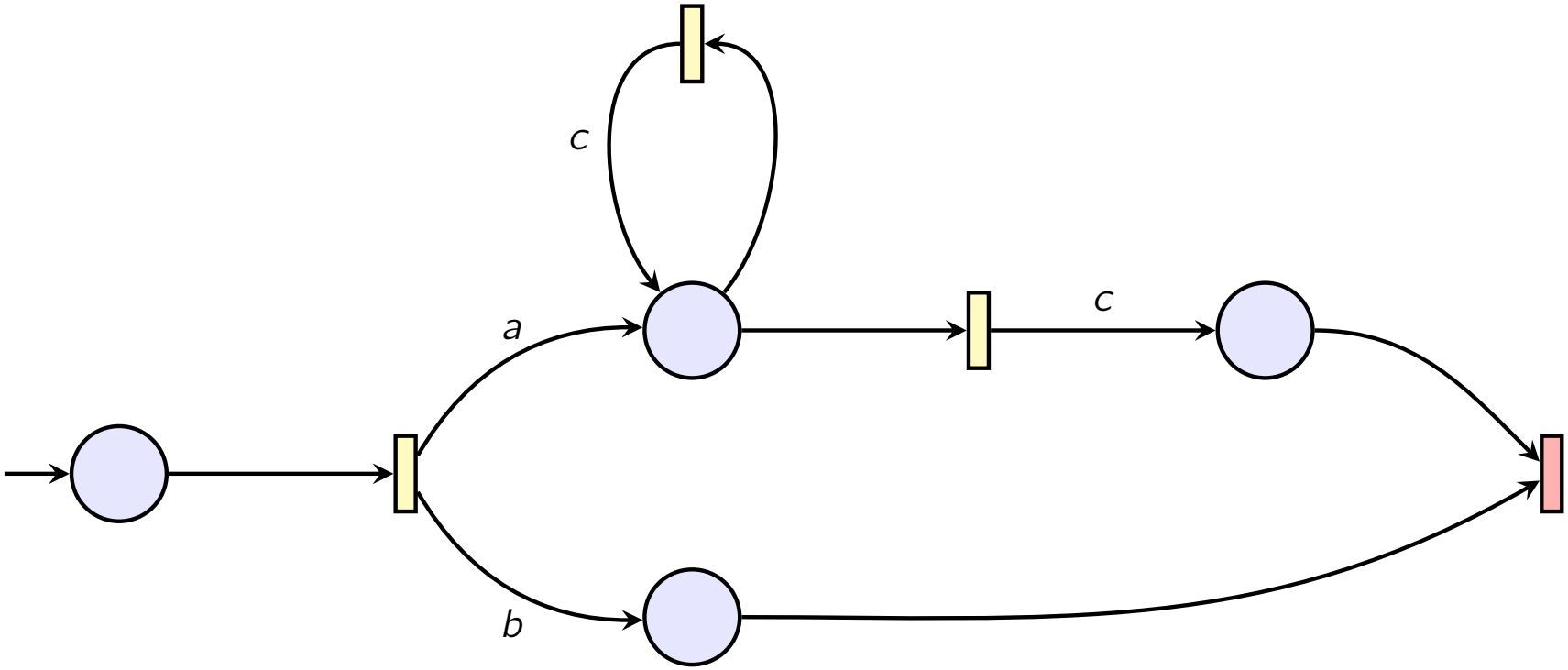
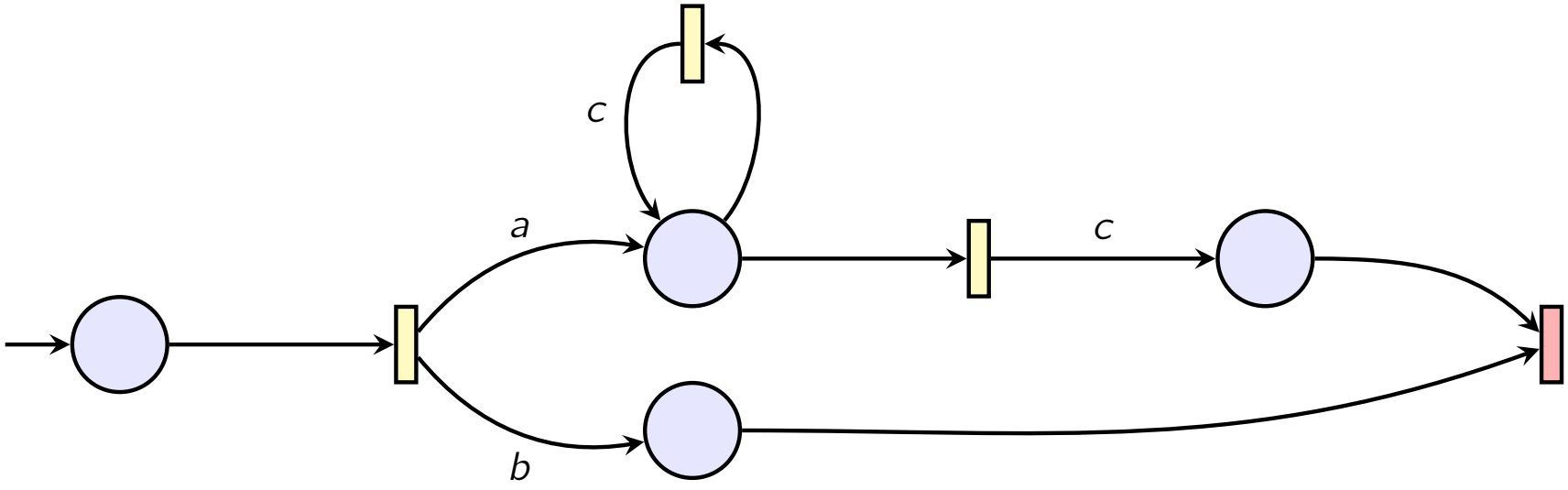
$$\mathcal{G}(0) = \emptyset \qquad \mathcal{G}(e + f) = \mathcal{G}(e) \cup \mathcal{G}(f) \qquad \mathcal{G}(e^{+}) = \bigcup_{n>0} \mathcal{G}(e)^{n}$$

Petri automata



Recognisable languages

Language of an automaton



Set of traces of $\mathcal{A}$

Recognisable languages

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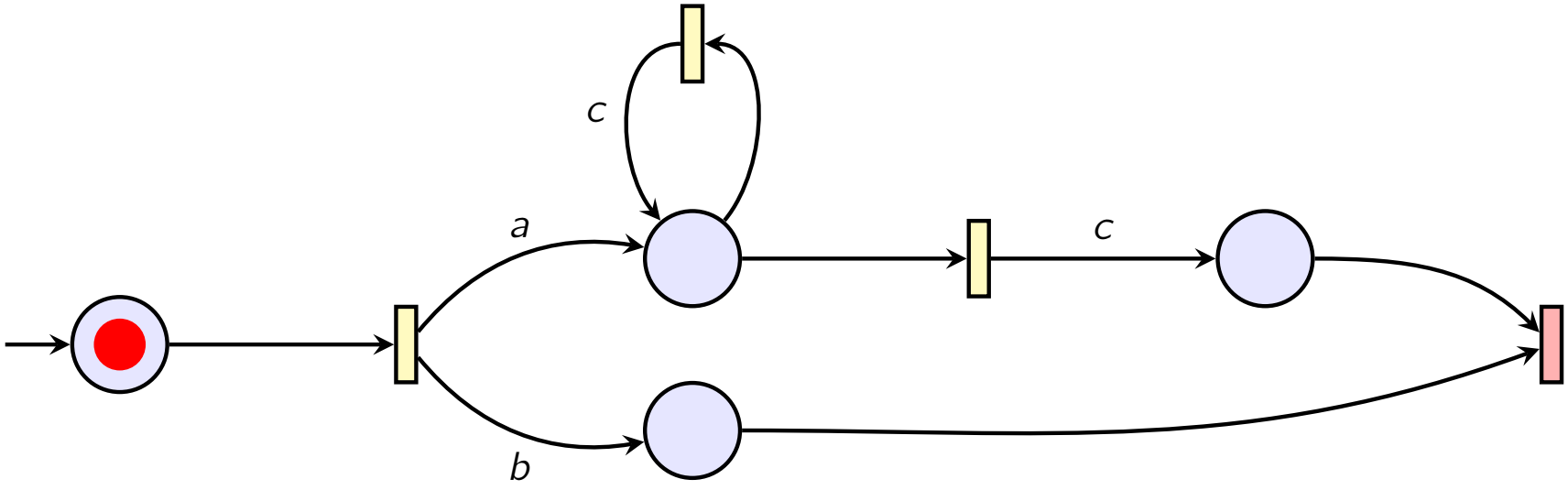

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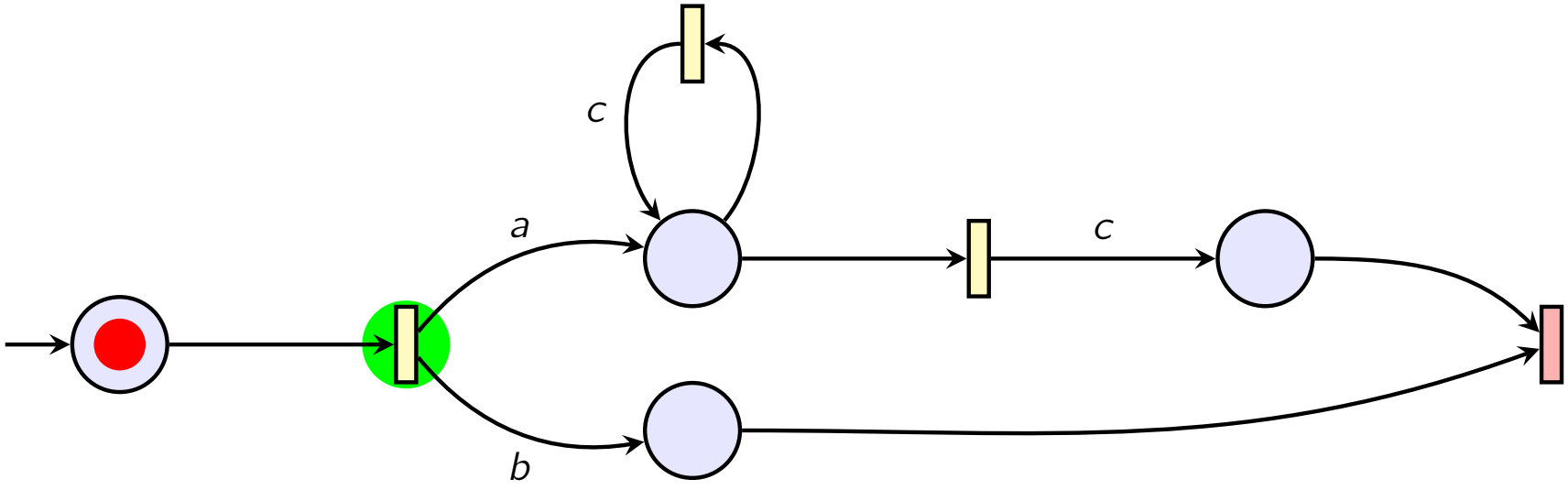

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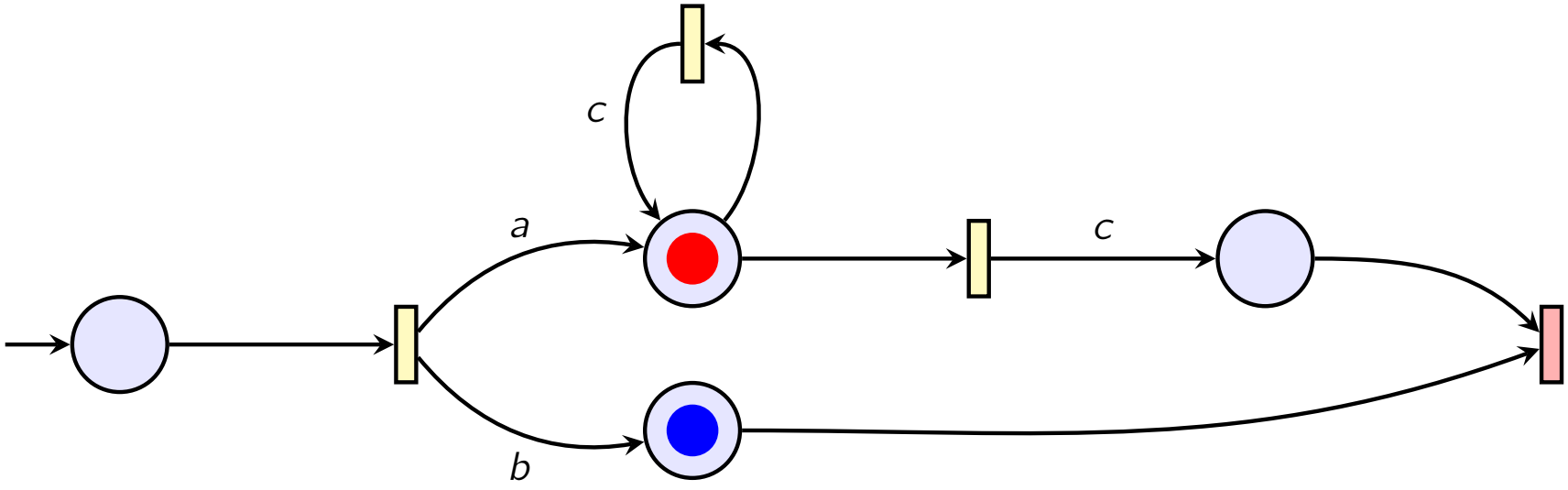

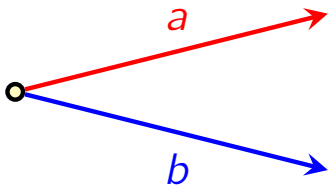

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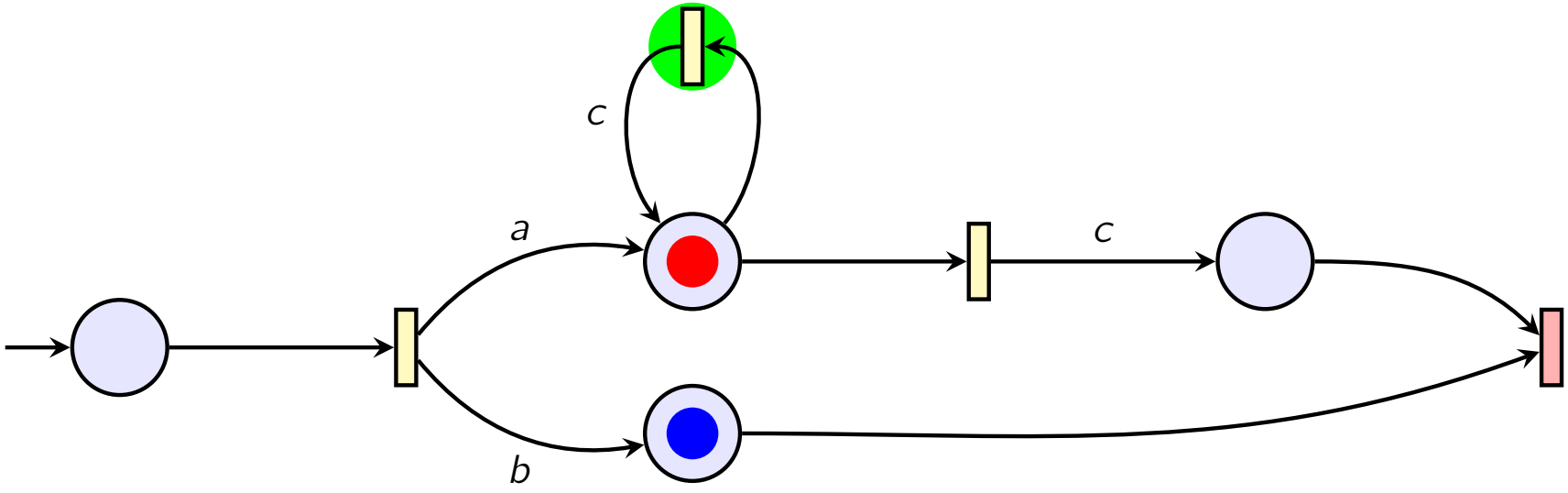

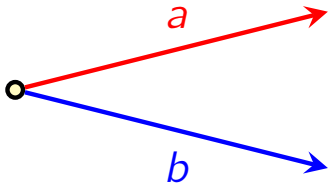

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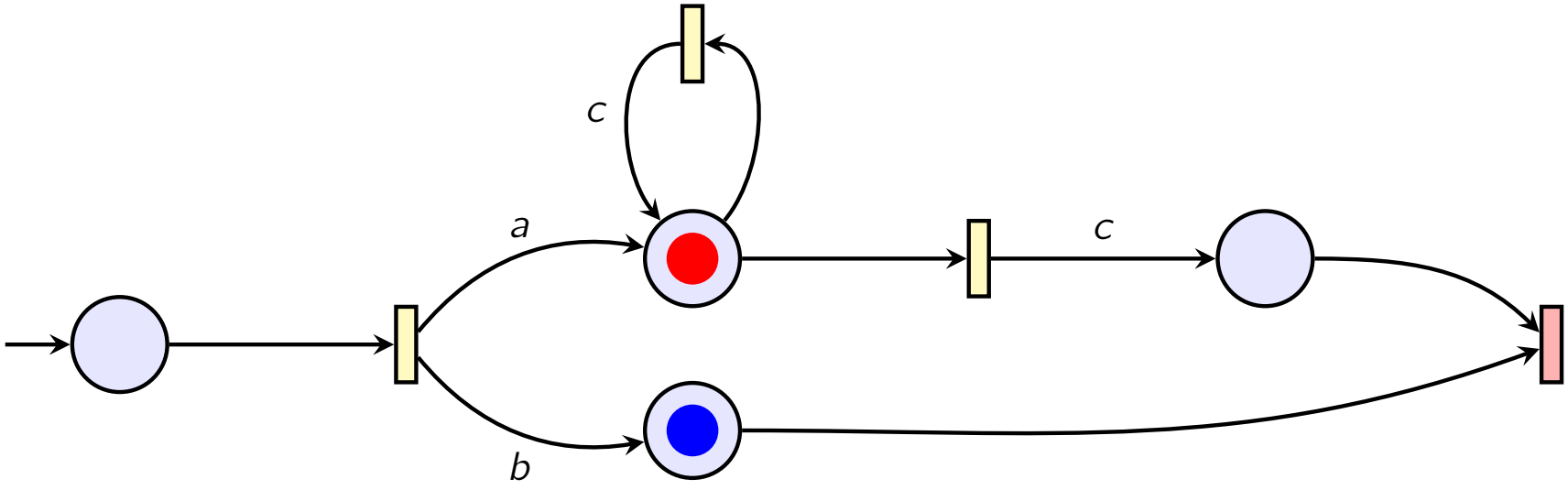

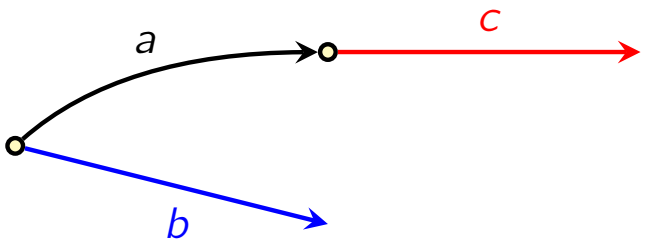

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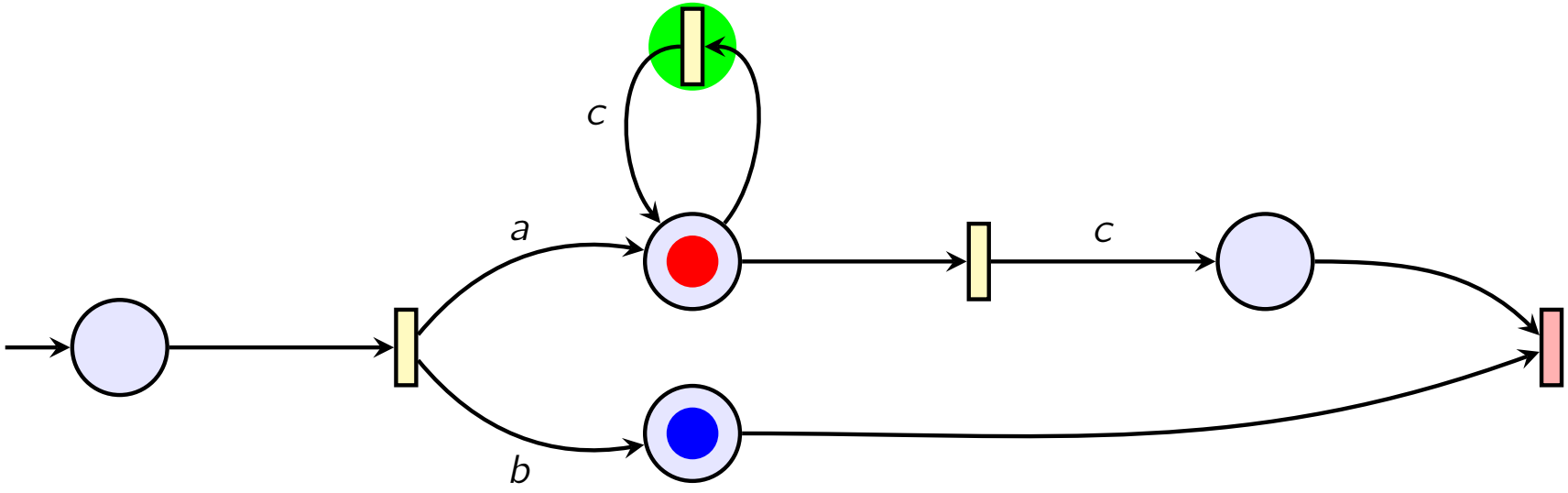

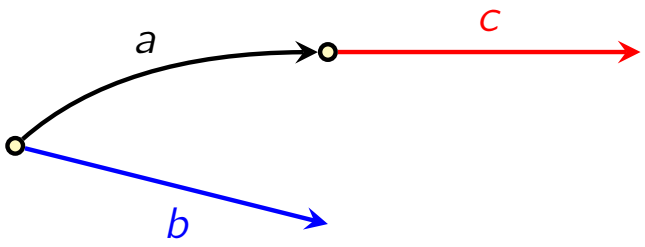

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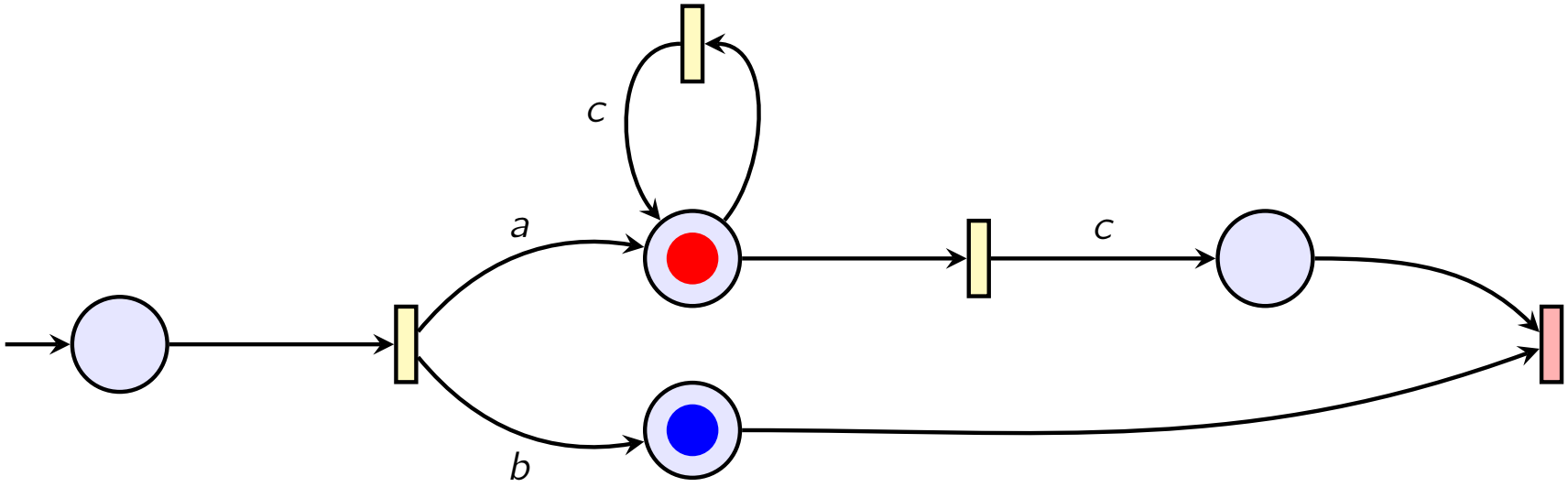

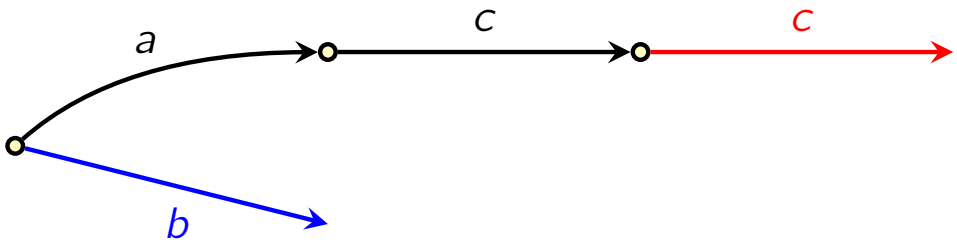

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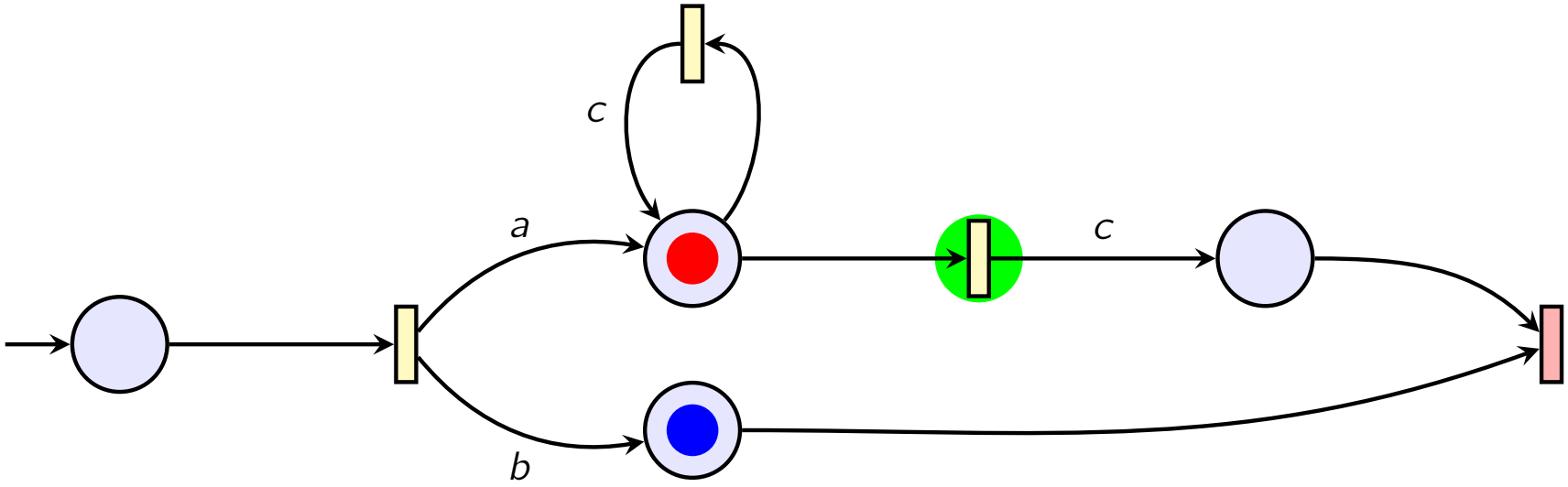

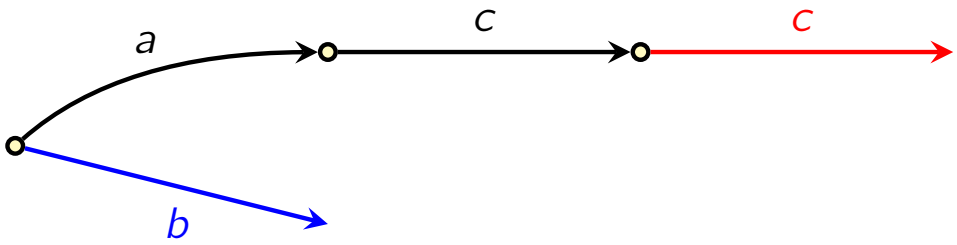

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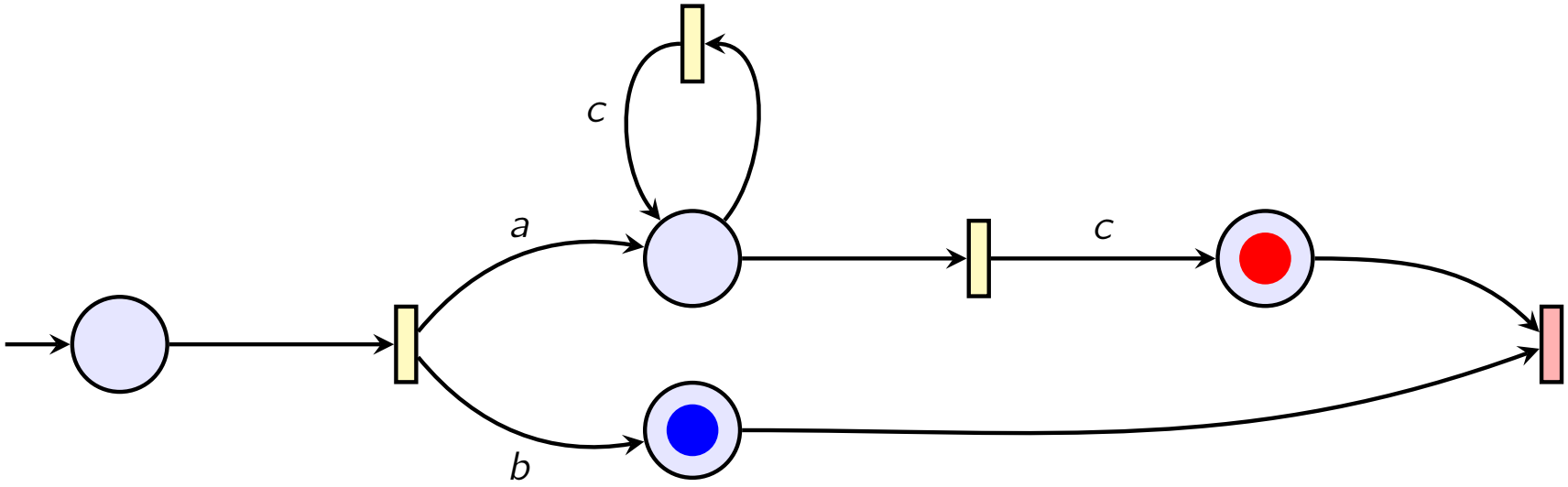

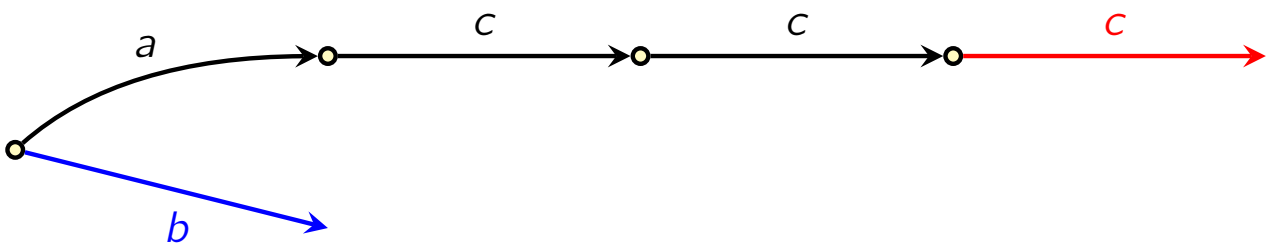

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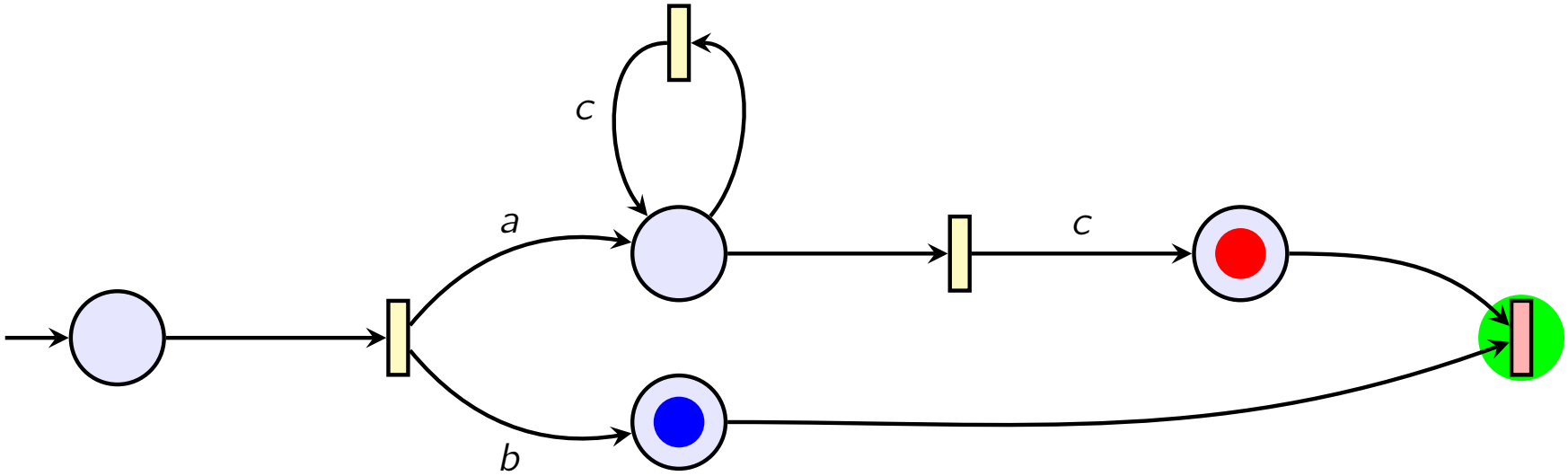

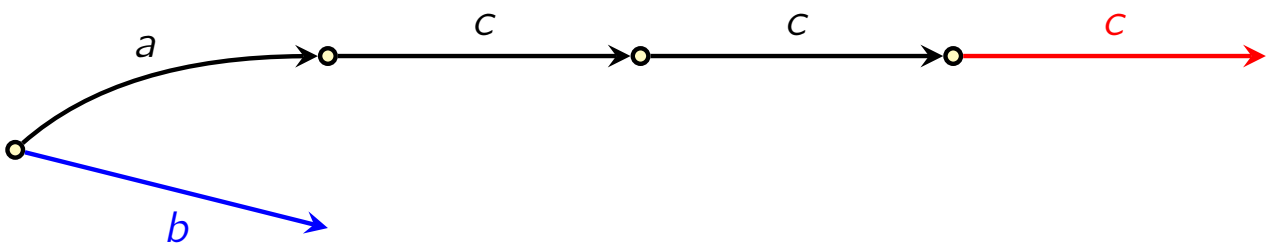

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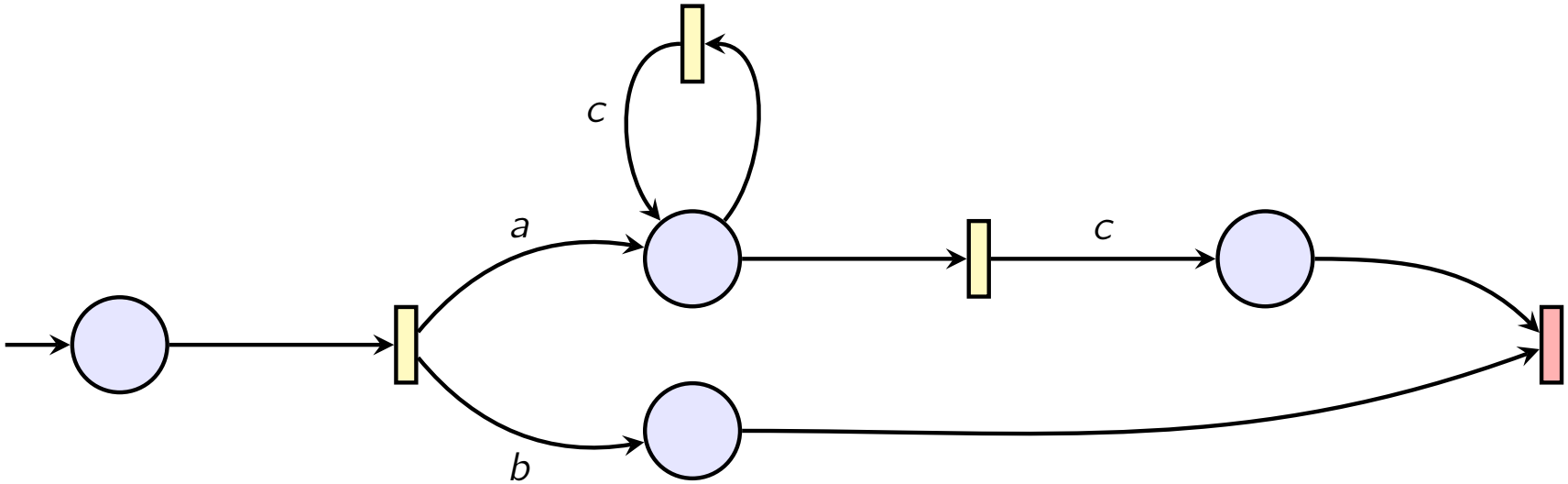

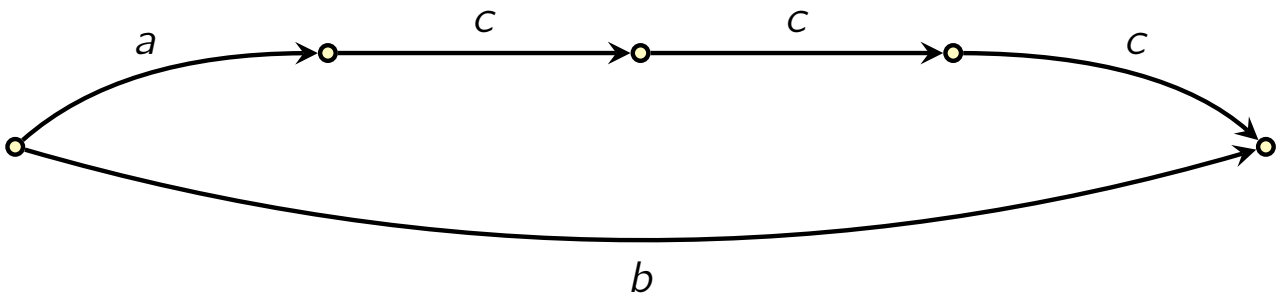

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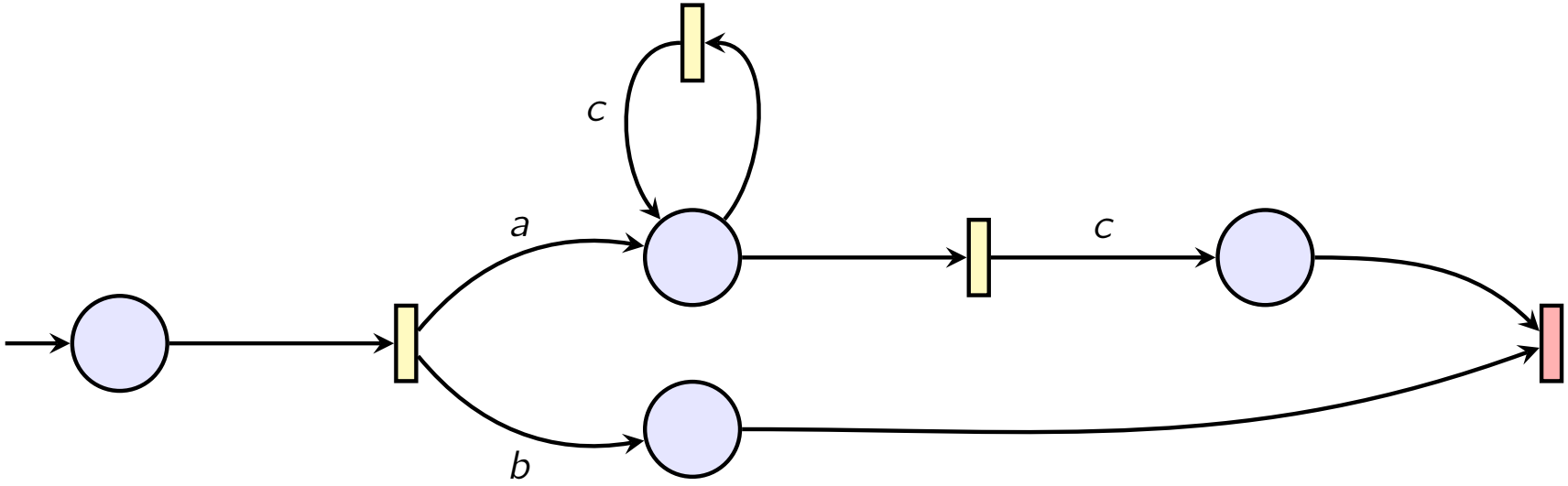
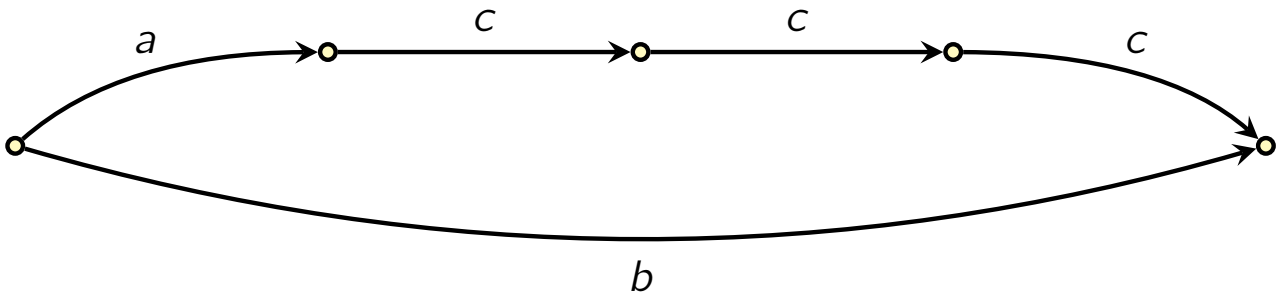


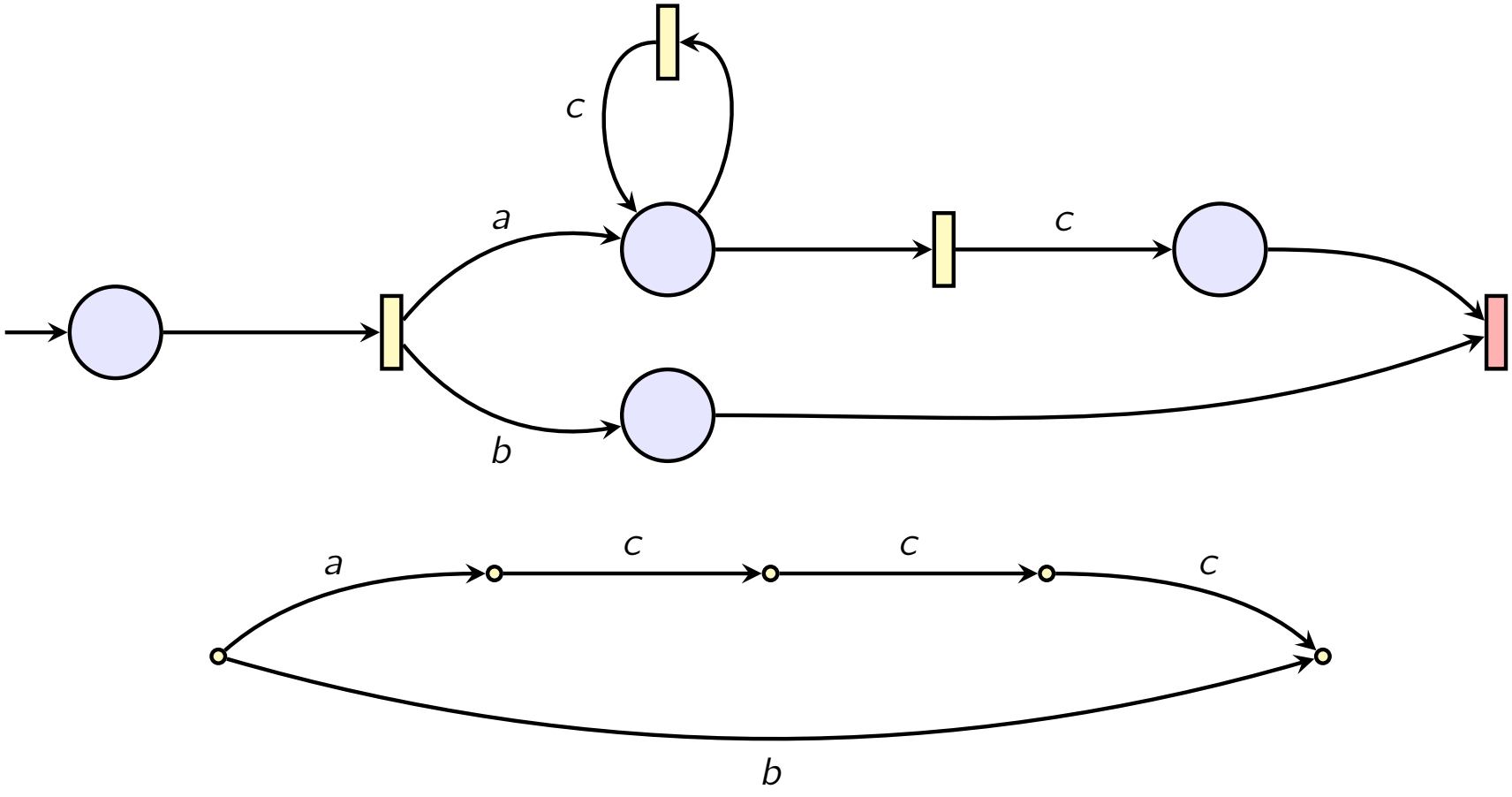


Set of traces of $\mathcal{A}$

$$\mathcal{T}r\,(\mathcal{A})$$

Recognisable languages

Language of an automaton



Set of traces of $\mathcal{A}$

$$\mathcal{T}r(\mathcal{A}) = \mathcal{G}((a \cdot c^{+}) \cap b).$$

Outline

Kleene Theorem

Theorem

Given a set of graphs X, the following are equivalent:

1. there is a **regular lattice expression** e such that $X = \mathcal{G}(e)$.
2. there is a **Petri automaton** $\mathcal{A}$ such that $X = \mathcal{T}r(\mathcal{A})$.

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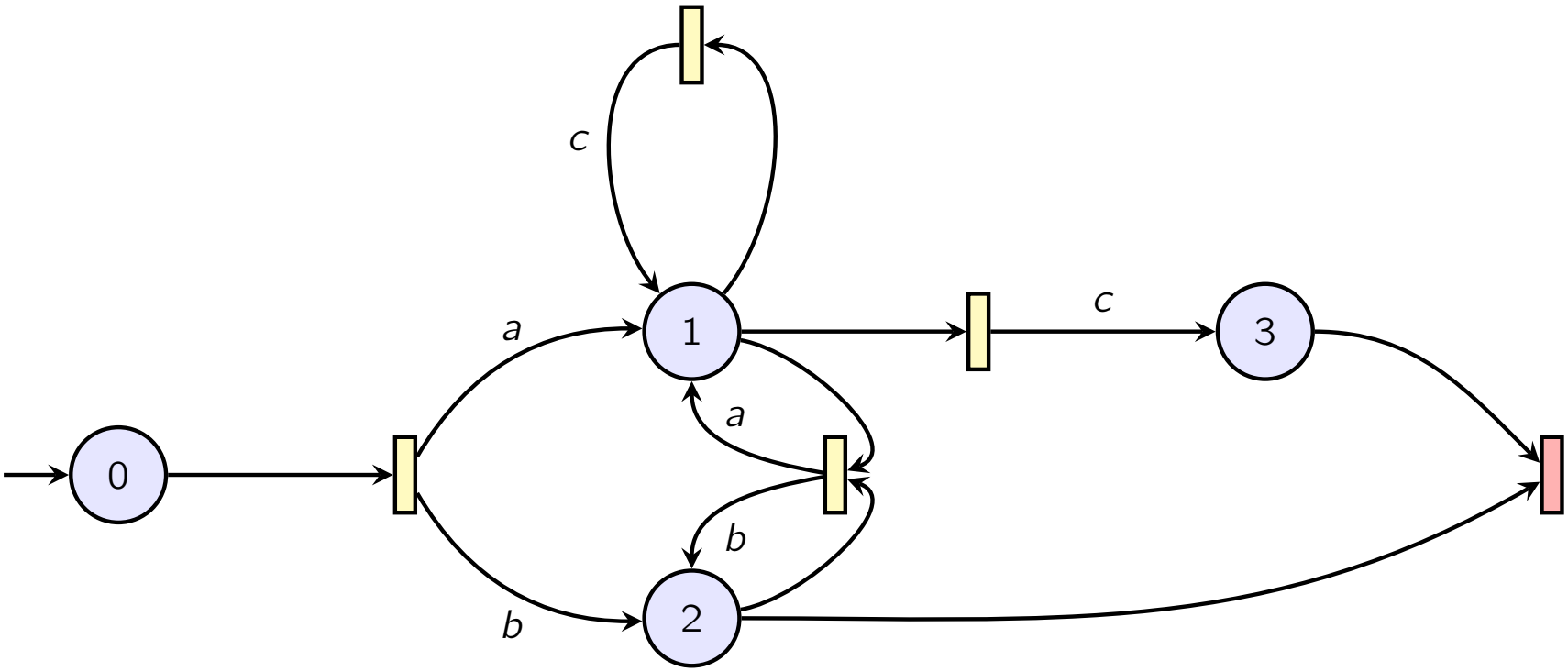
Proof.

- $1 \Rightarrow 2$: inductive construction;
  B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*
- $2 \Rightarrow 1$: this talk

□

From automata to expressions



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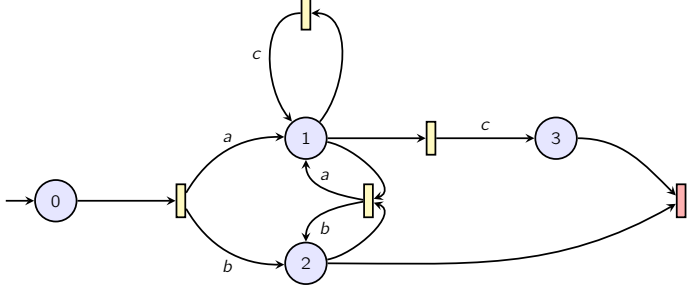

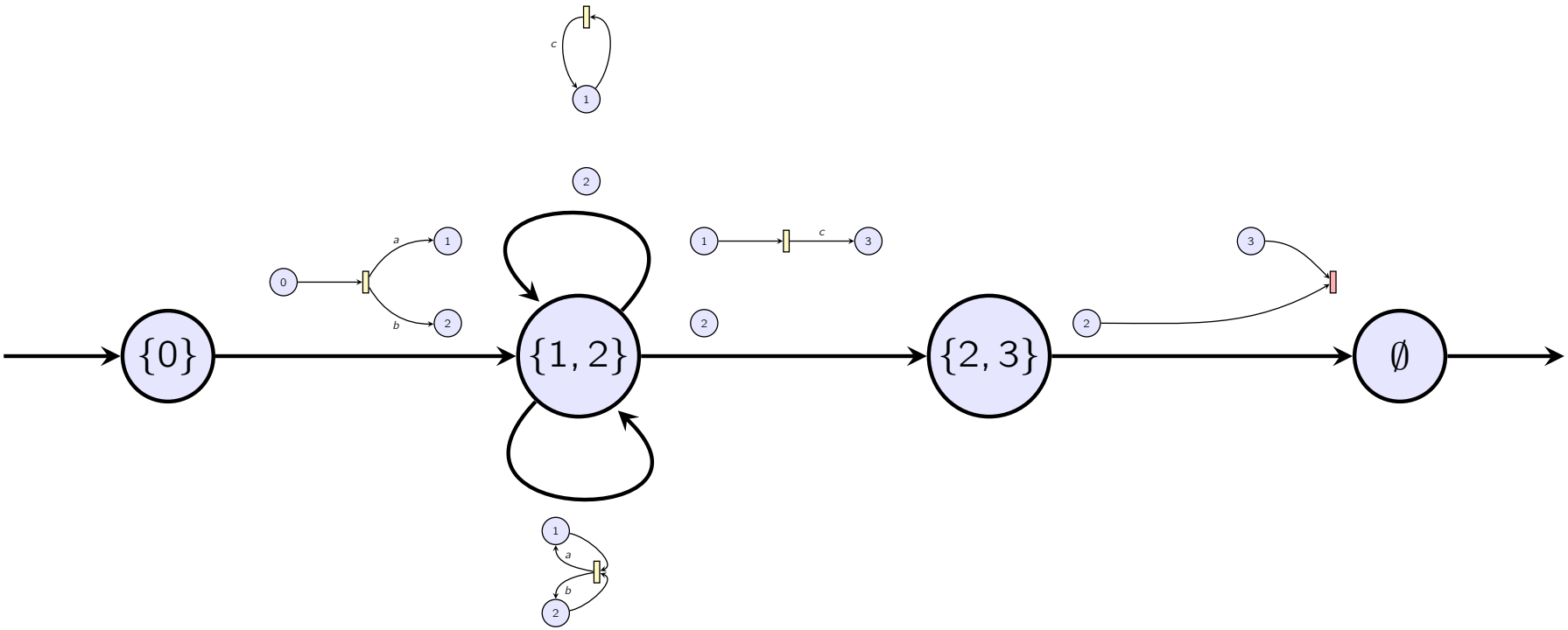

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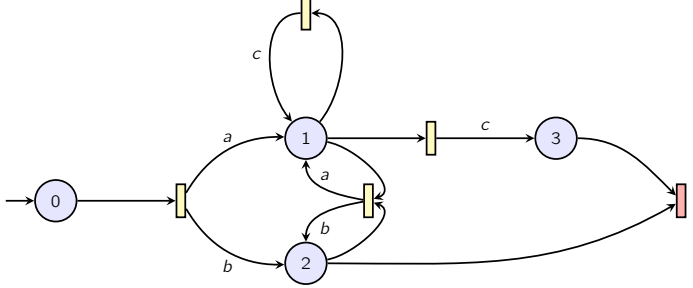

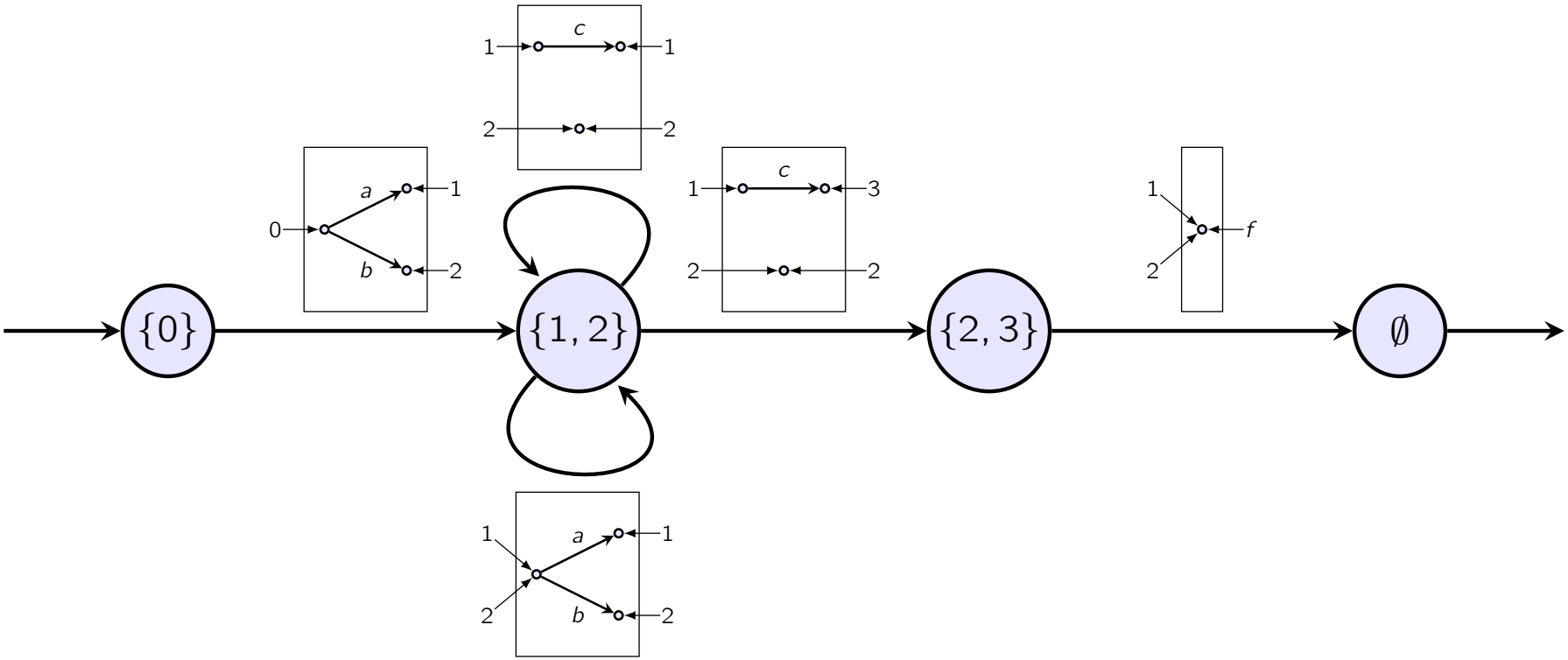

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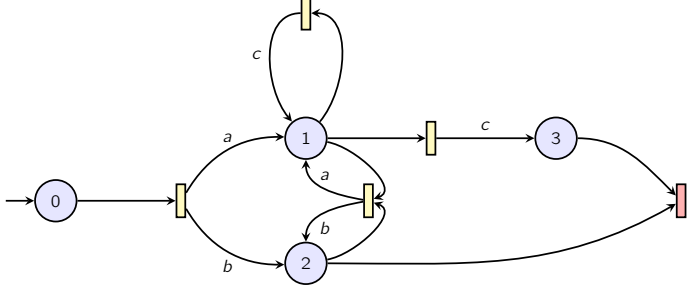

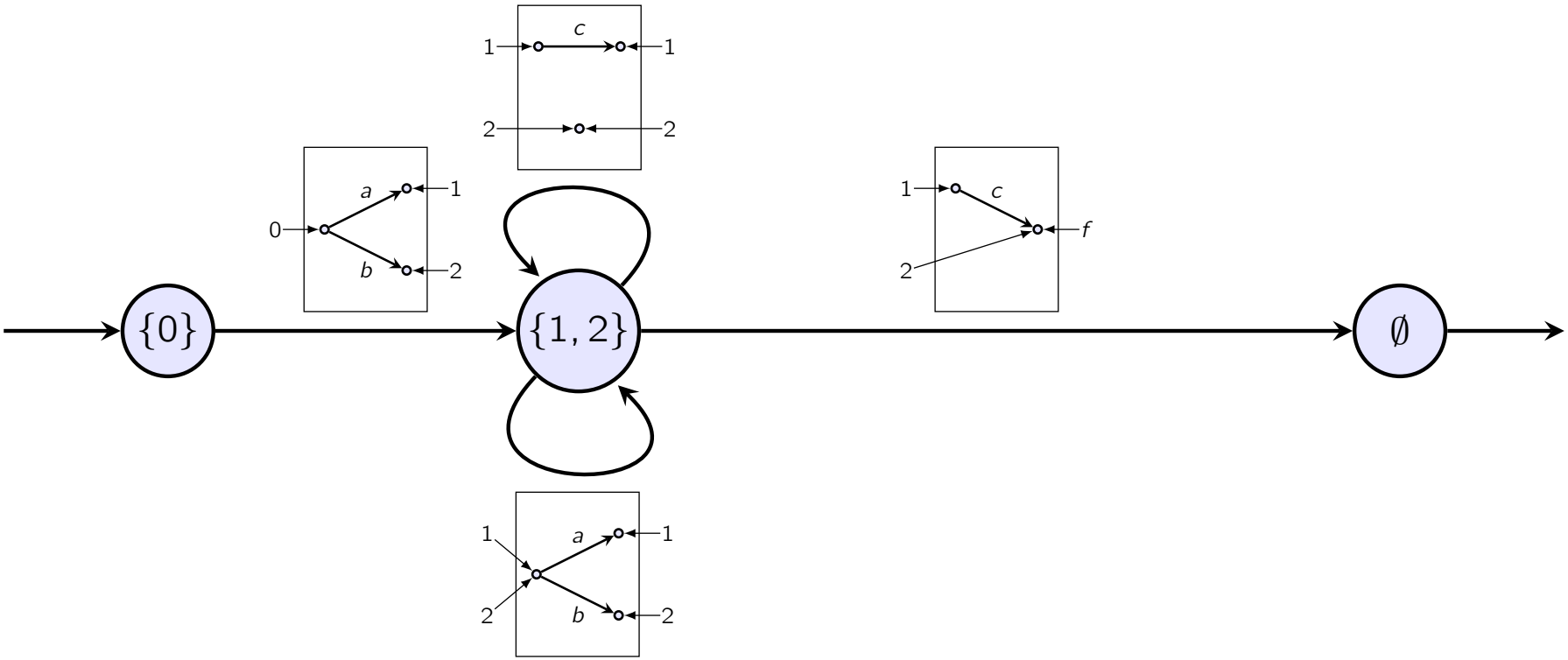

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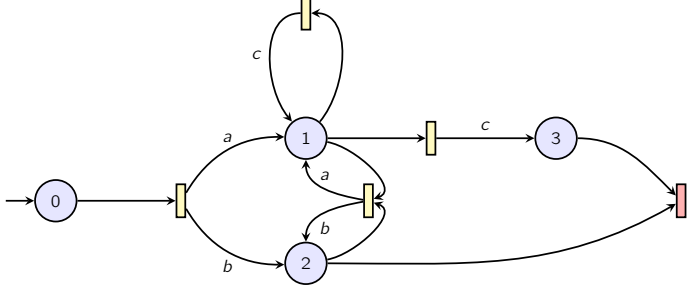

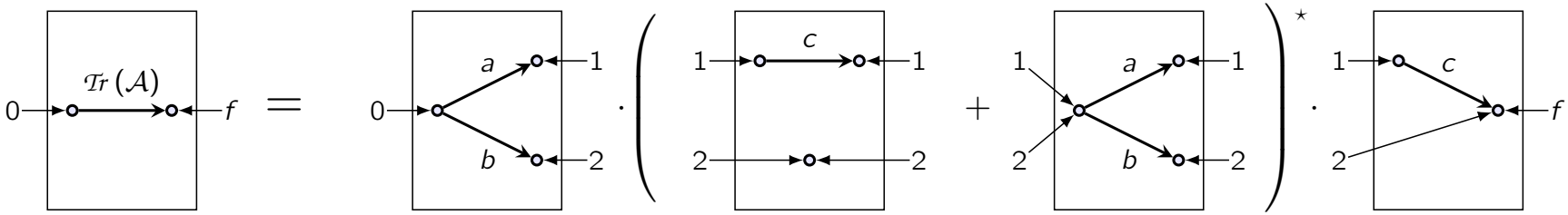

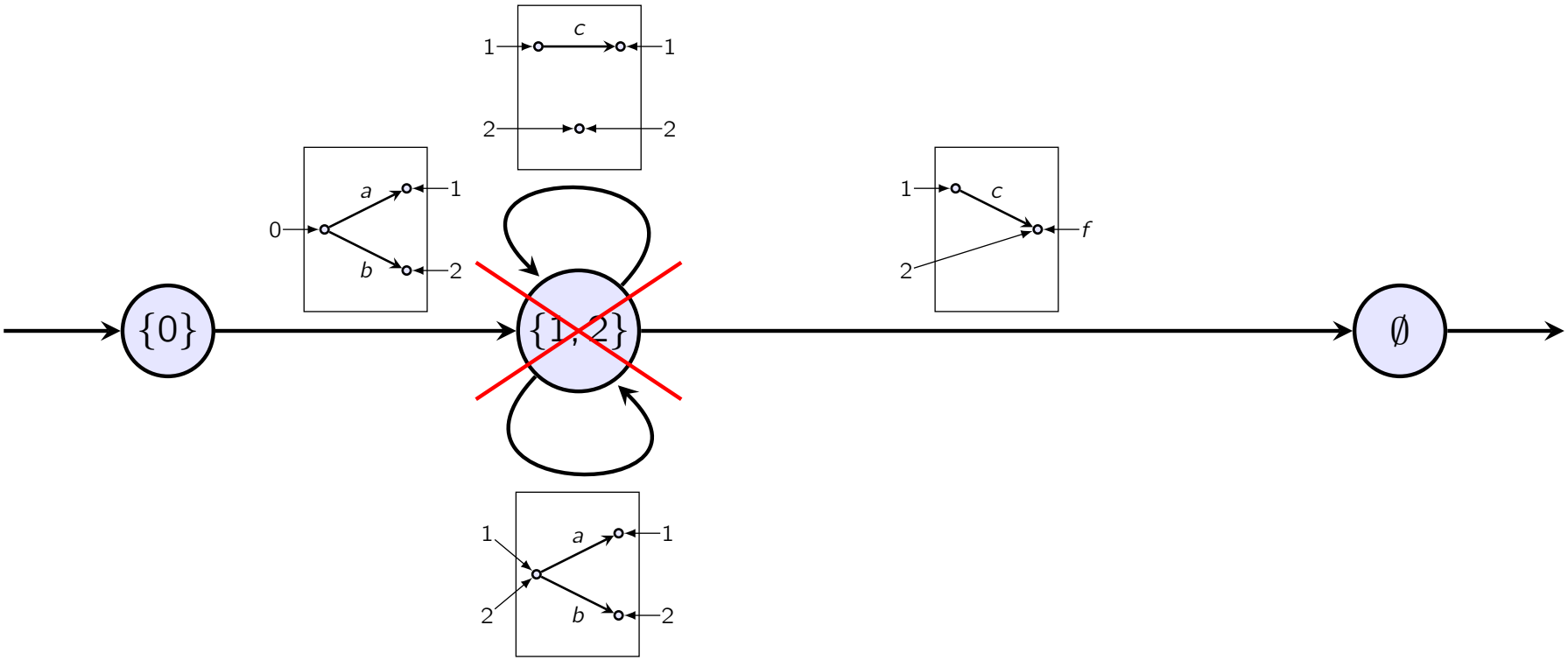

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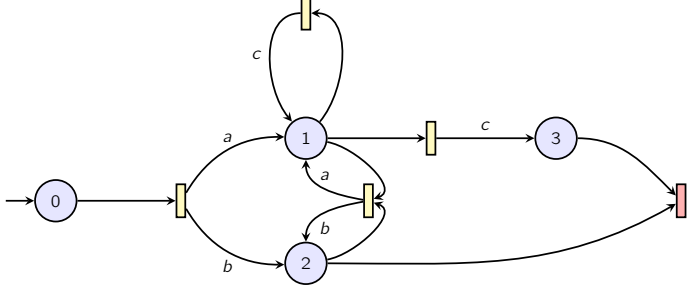
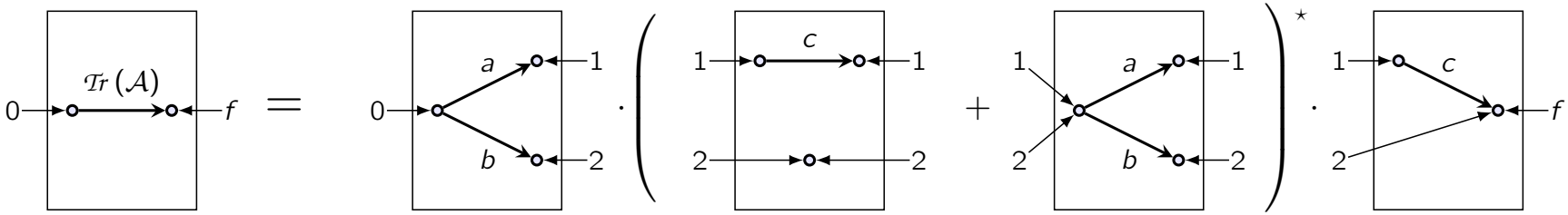
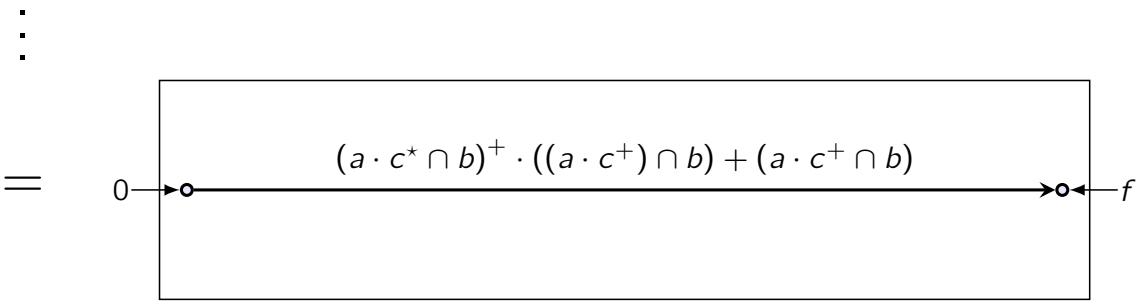
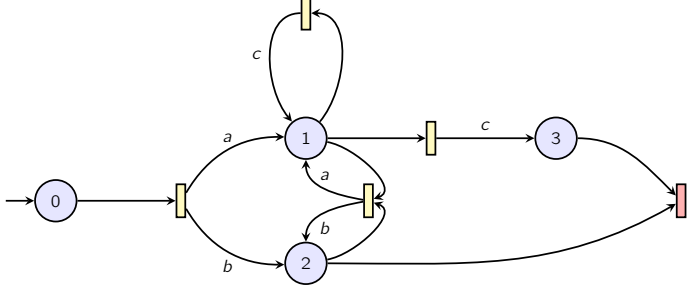


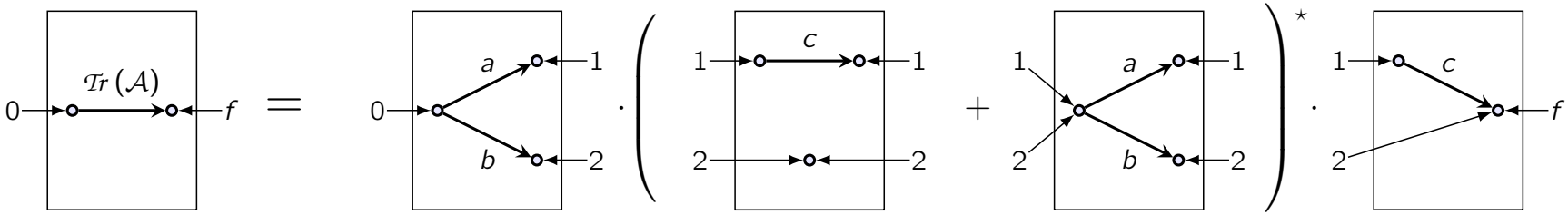


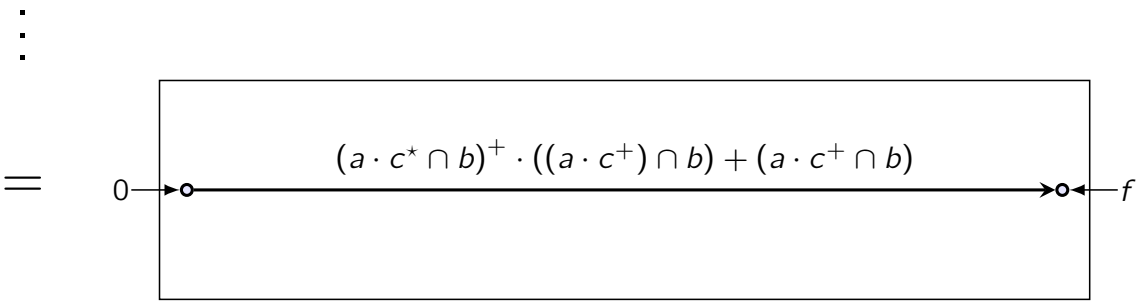


(With some effort...)

From automata to expressions







Result

$$\mathcal{T}r(\mathcal{A}) = \mathcal{G}\left((a \cdot c^\star \cap b)^+ \cdot \left(\left(a \cdot c^+\right) \cap b\right) + \left(a \cdot c^+ \cap b\right)\right)$$

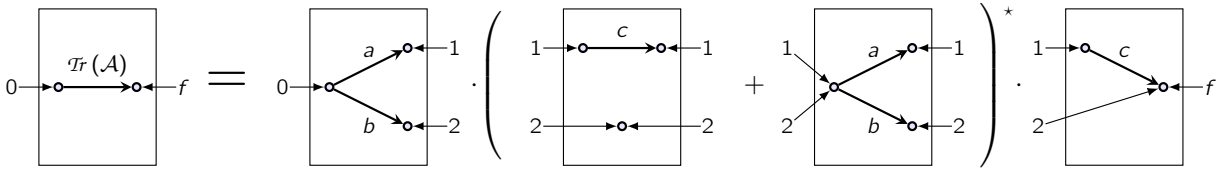
That's all folks!

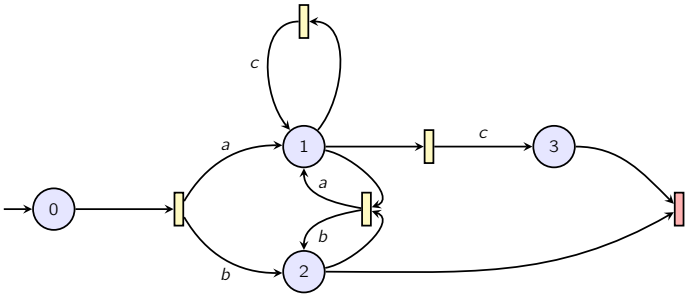
Thank you!

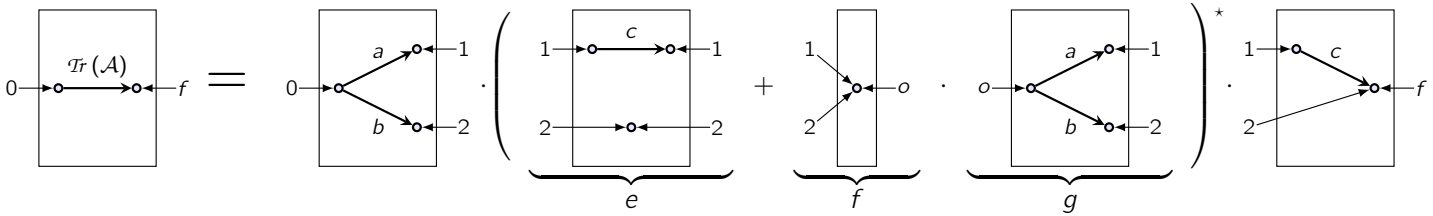
See more at:
`http://perso.ens-lyon.fr/paul.brunet`

Outline

From automata to expressions







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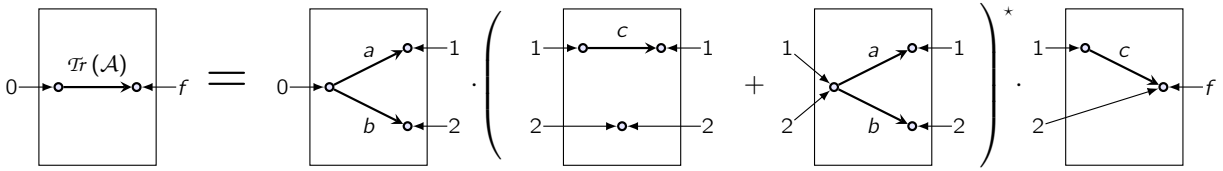
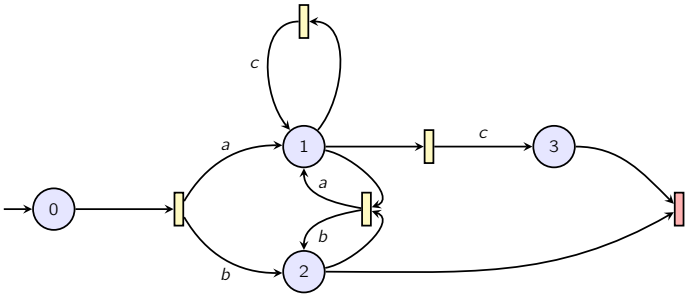
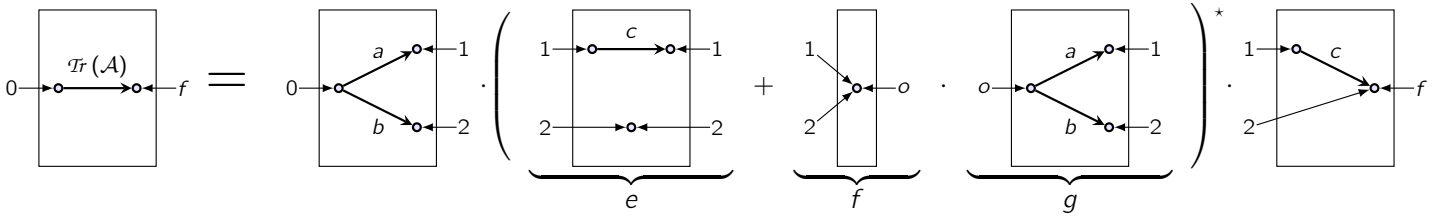
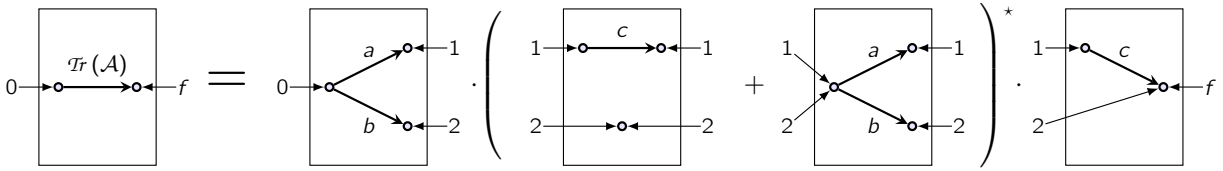


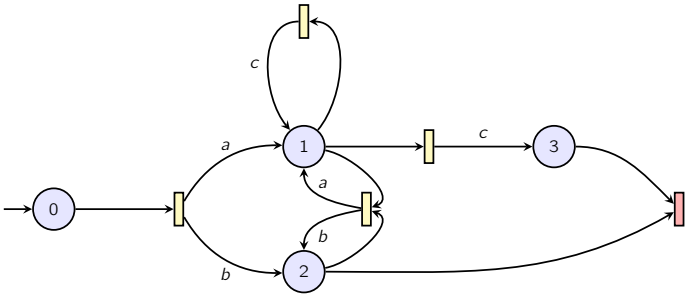


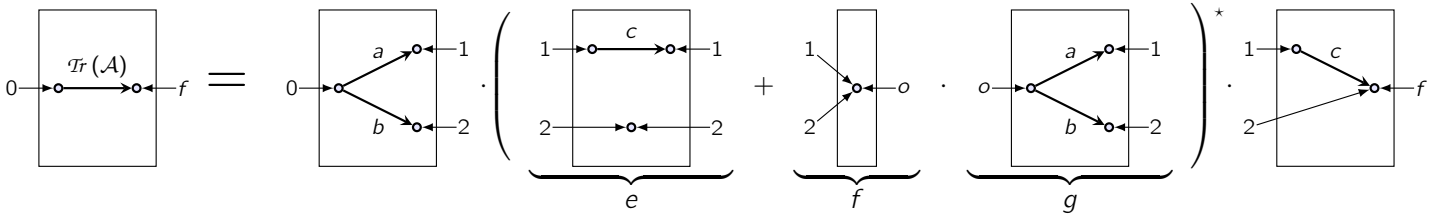


$$
\begin{aligned}
(e+fg)^\star &= e^\star (fge^\star)^\star \\
&= e^\star \left(1 + fge^\star (fge^\star)^\star\right) \\
&= e^\star \left(1 + f (ge^\star f)^\star ge^\star\right) \\
&= e^\star + e^\star f (ge^\star f)^\star ge^\star
\end{aligned}
$$

From automata to expressions







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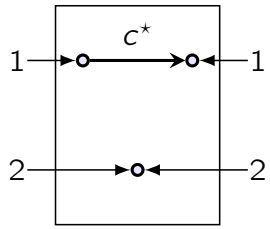

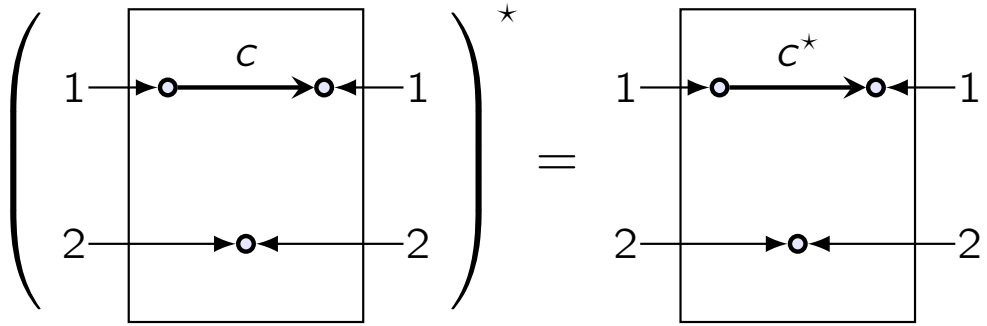

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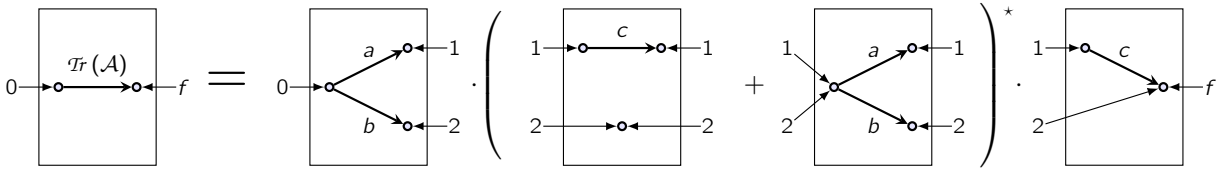

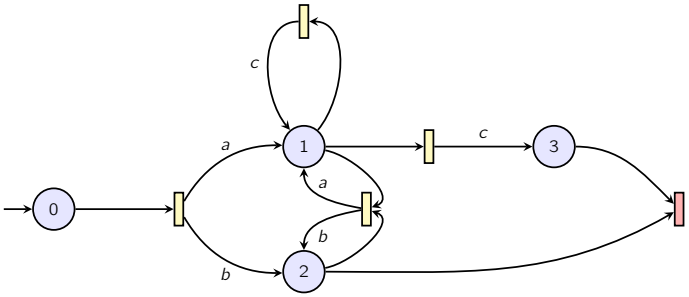

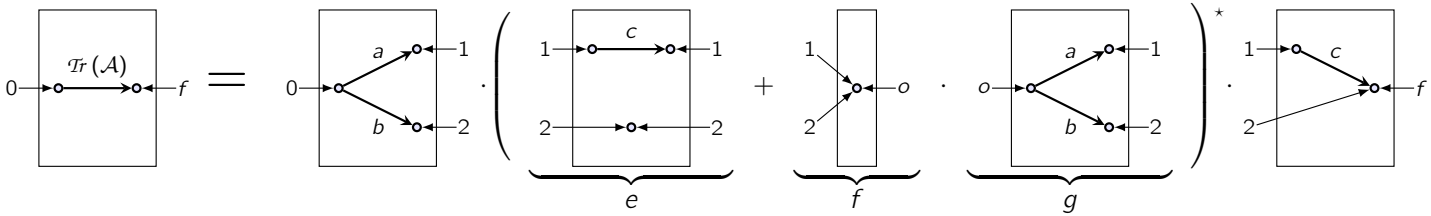

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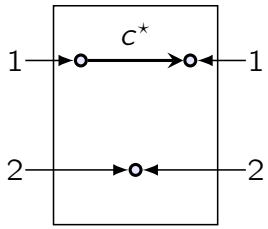

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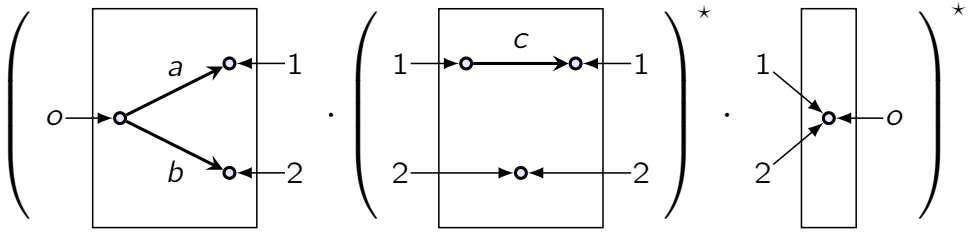

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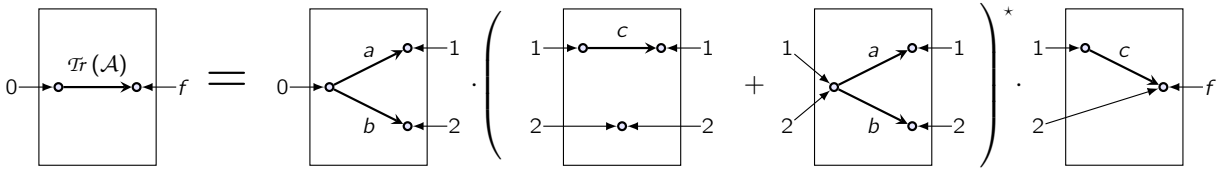

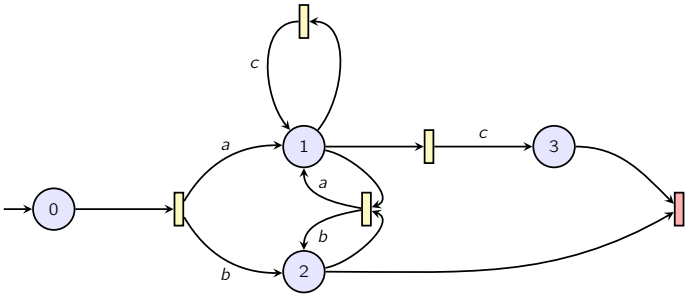

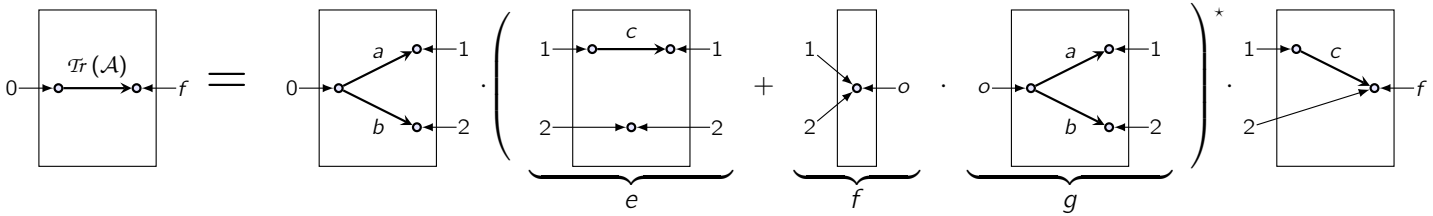

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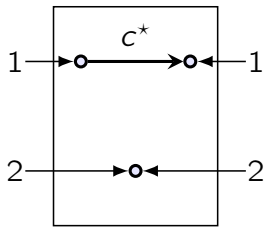

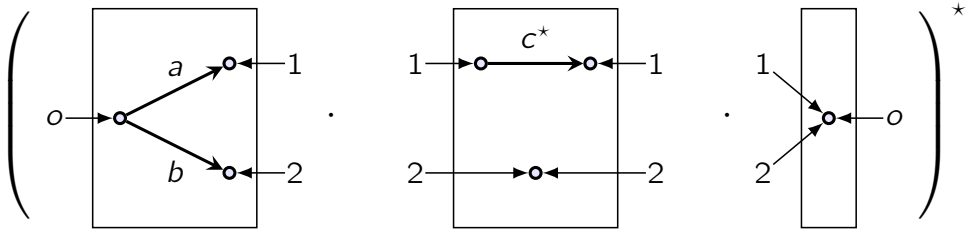

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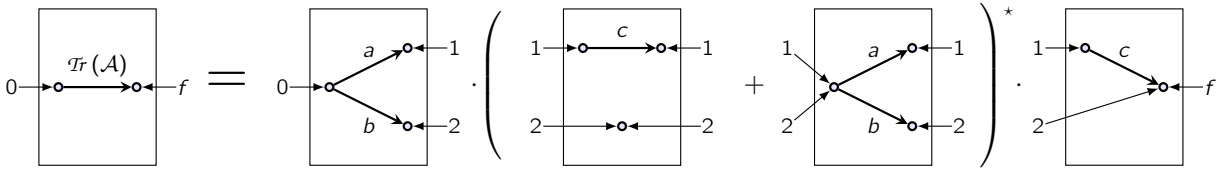

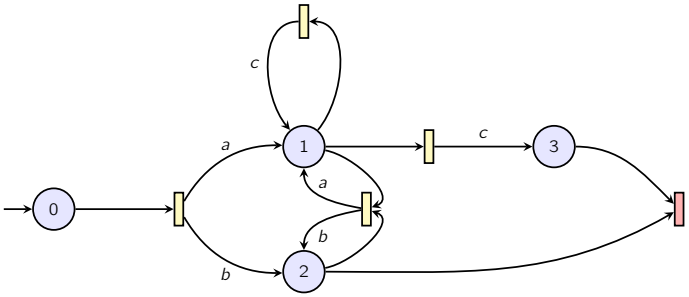

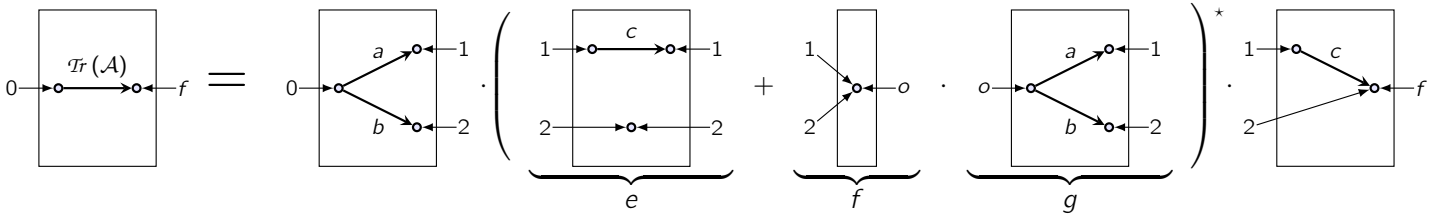

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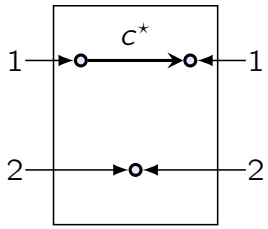

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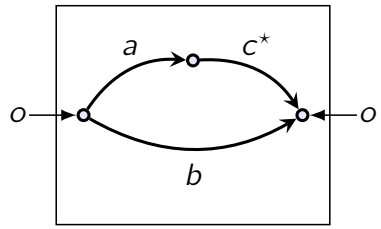

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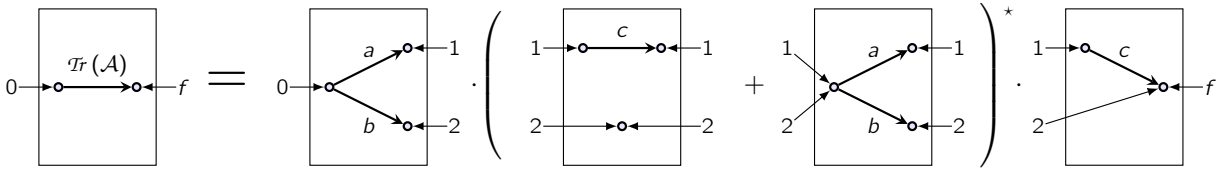

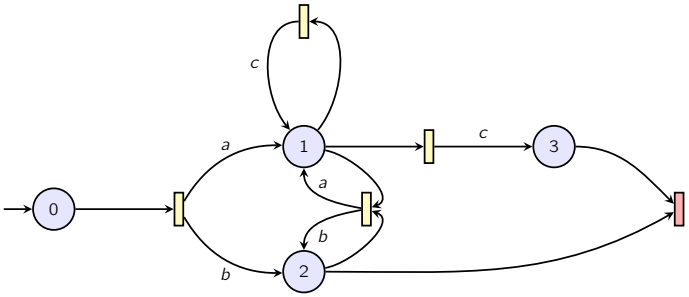

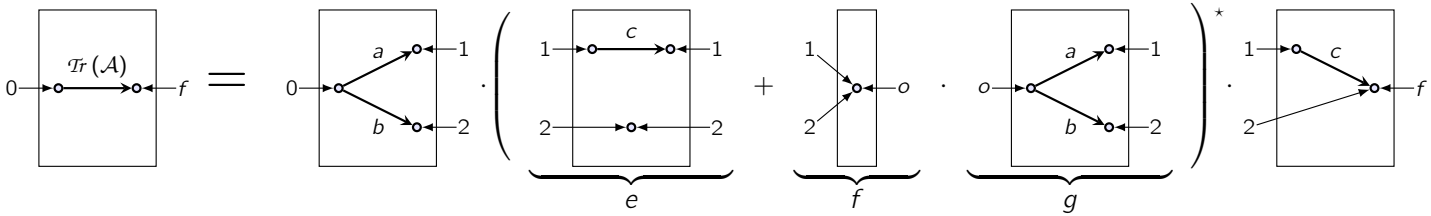

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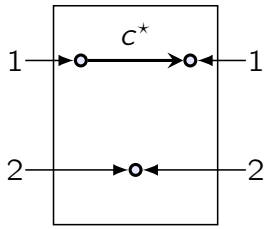

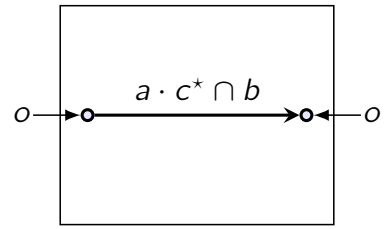

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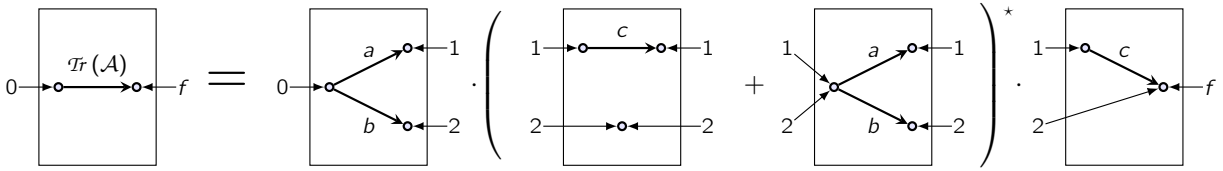

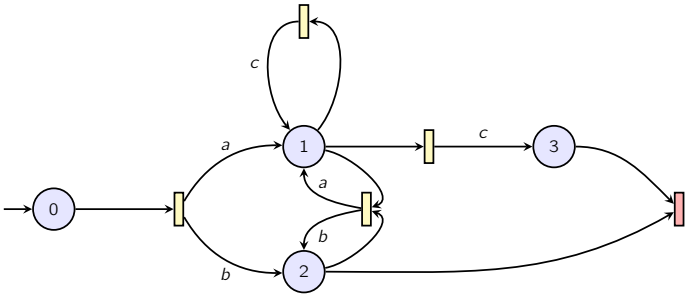

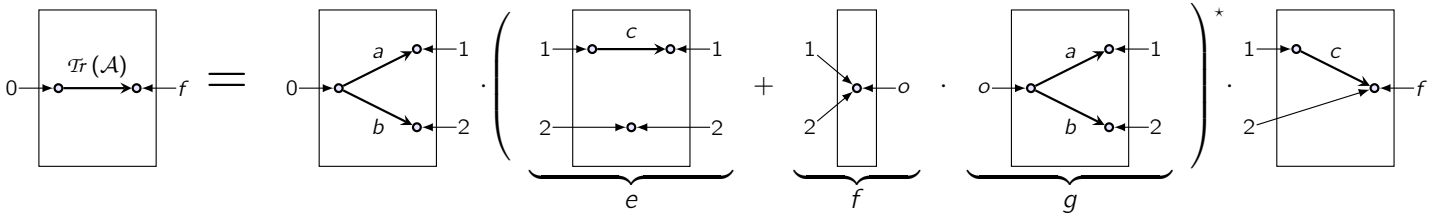

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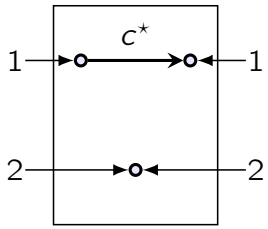

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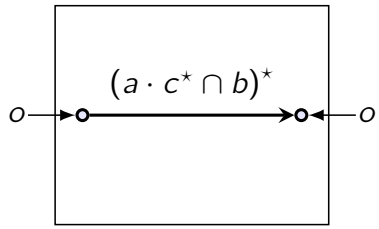

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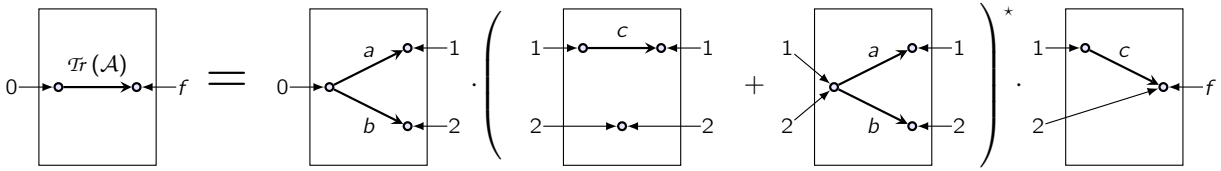

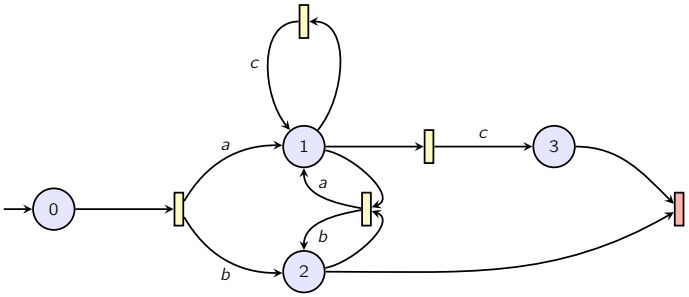

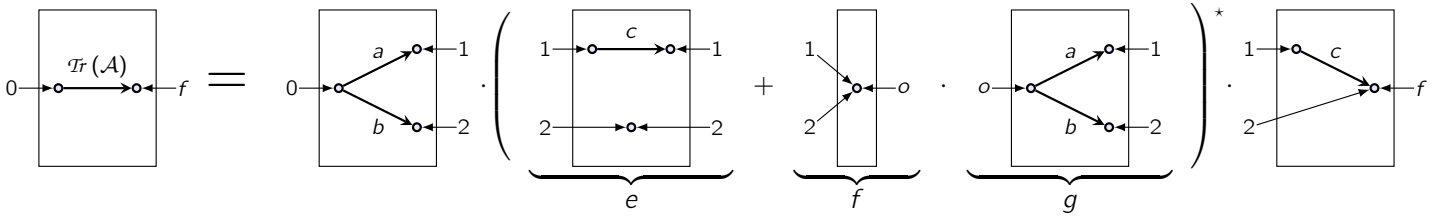

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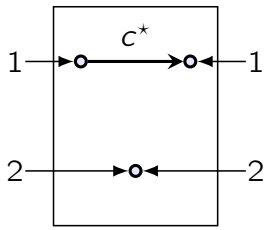

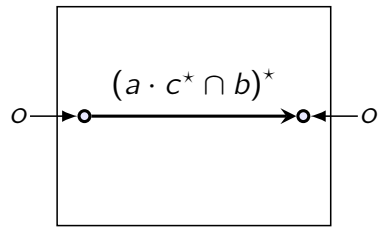

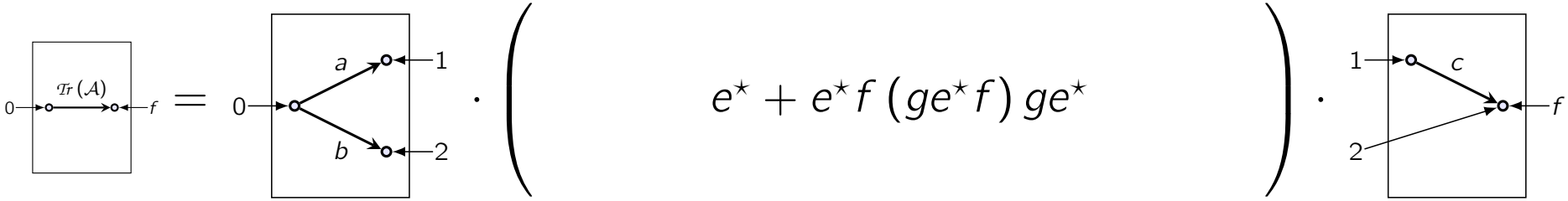

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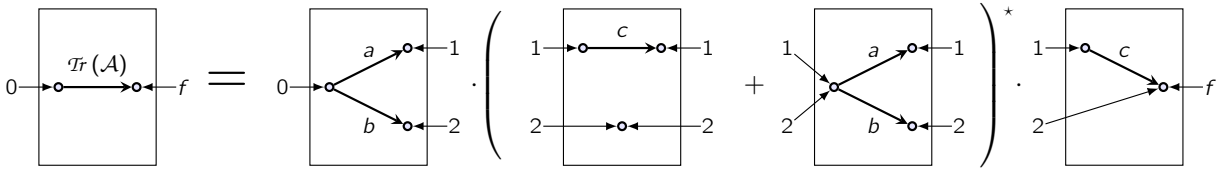

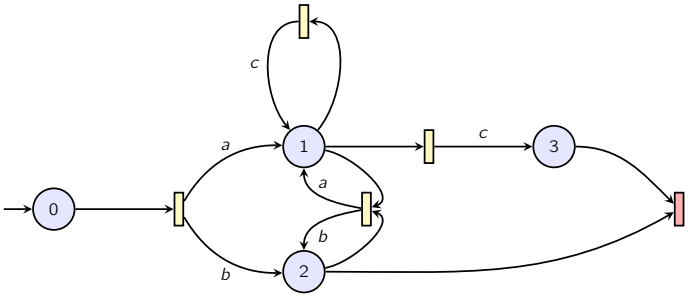

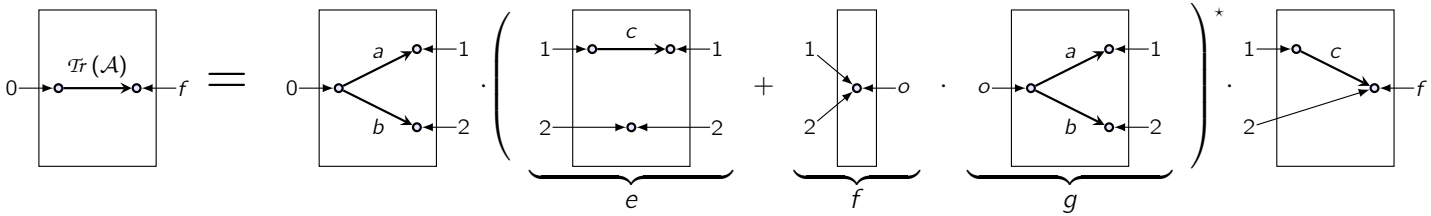

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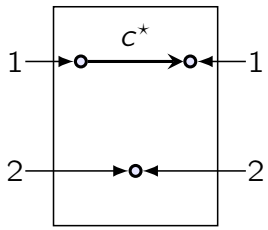

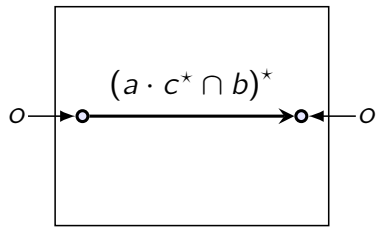

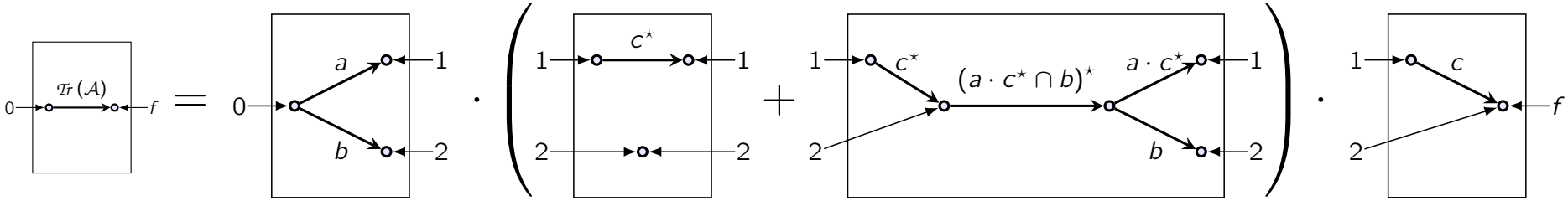

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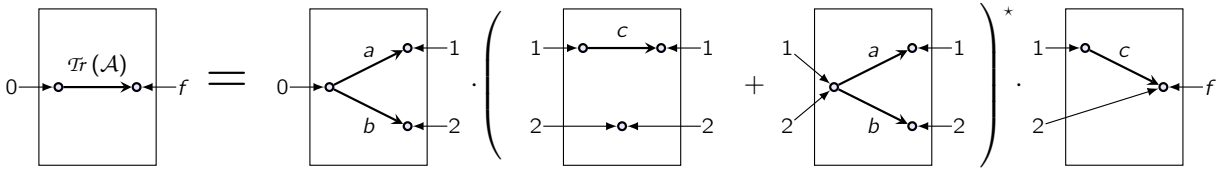

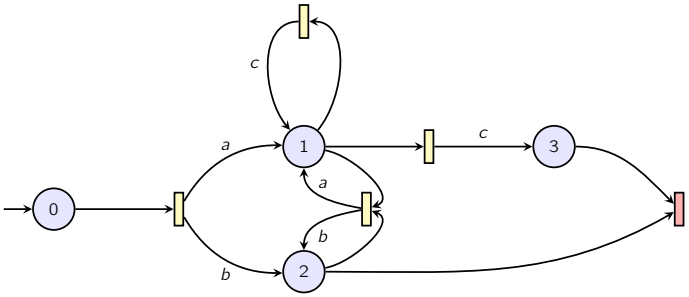

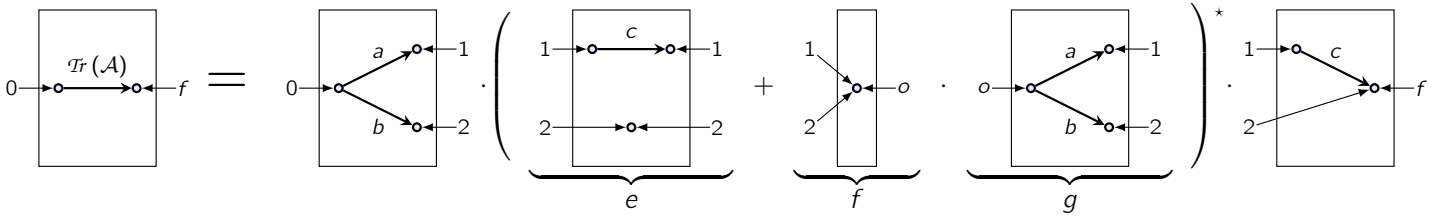

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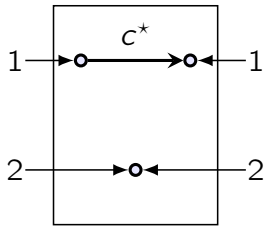

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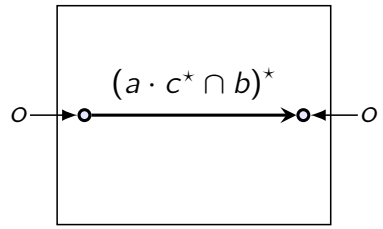

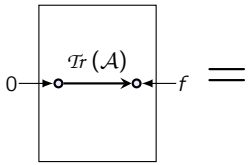

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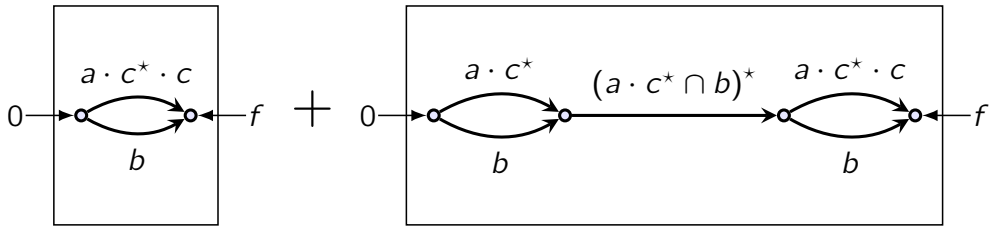

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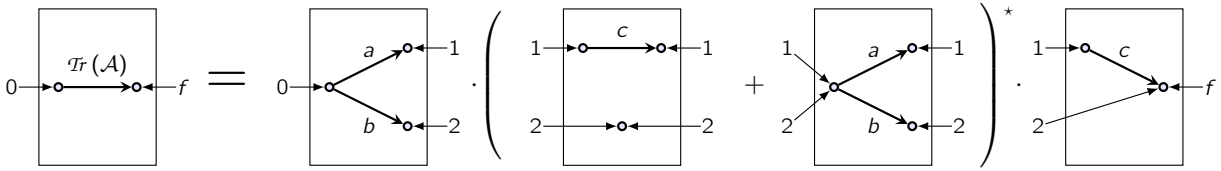

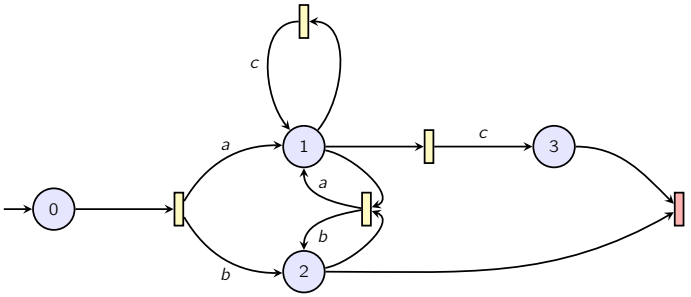

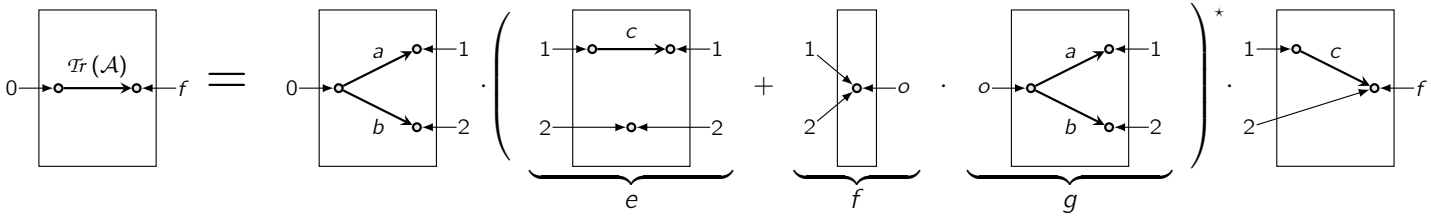

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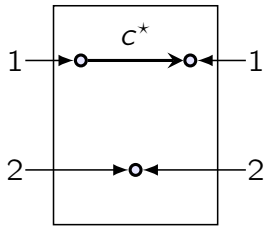

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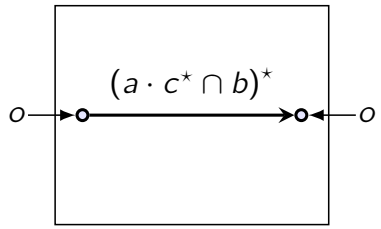

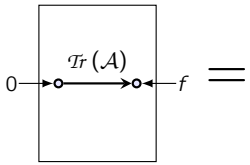

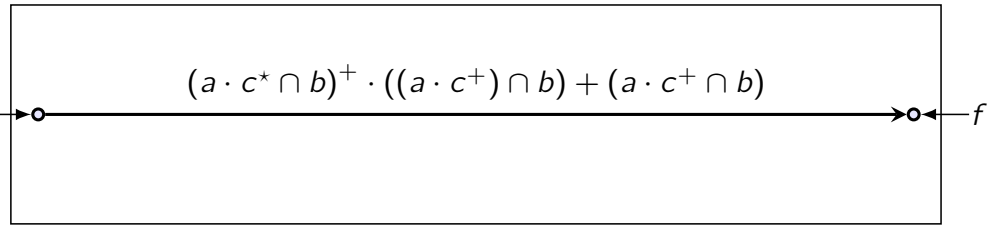

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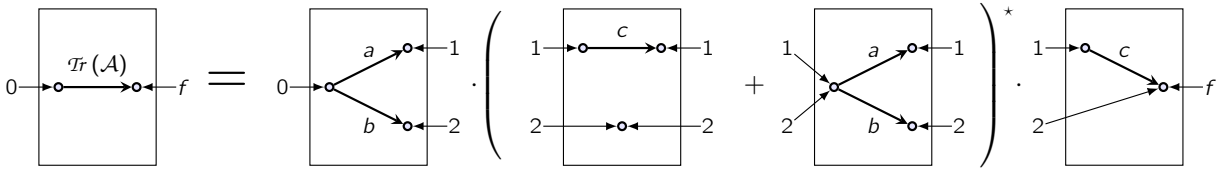
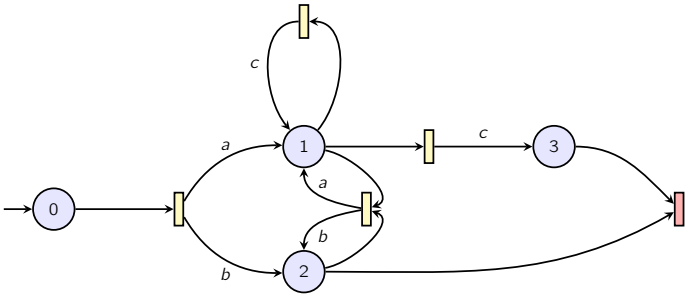
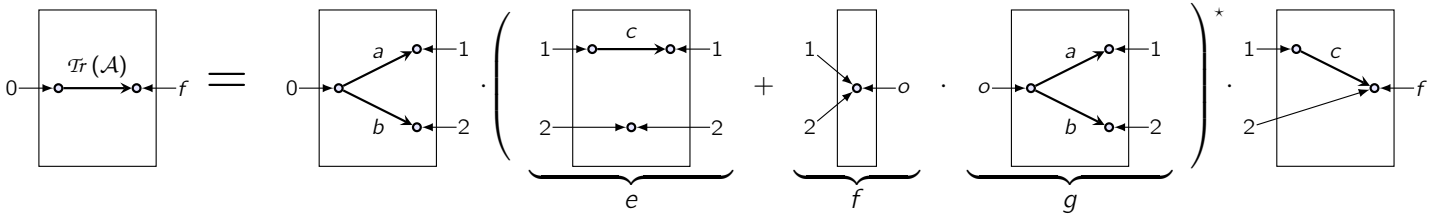
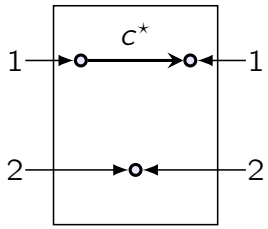




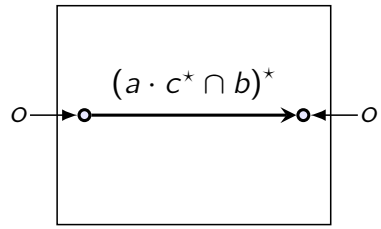


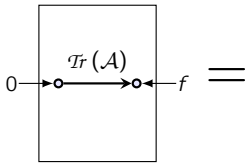
$$(e + fg)^\star = e^\star + e^\star f\,(ge^\star f)^\star ge^\star$$

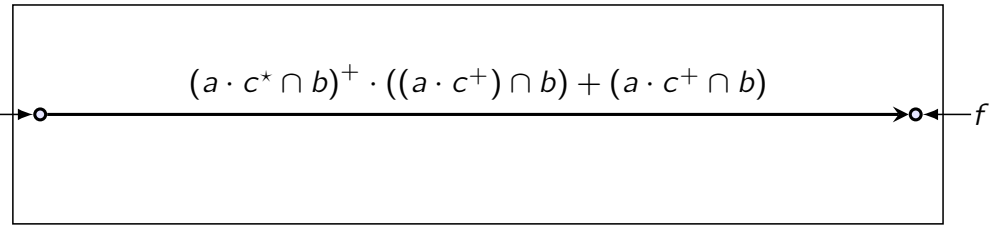
$e^\star =$



$(ge^\star f)^\star =$







Result

$$\mathcal{Tr}(\mathcal{A}) = \mathcal{G}\left((a \cdot c^\star \cap b)^+ \cdot \left(\left(a \cdot c^+\right) \cap b\right) + \left(a \cdot c^+ \cap b\right)\right)$$