Relation Algebra

from algorithms to formal proofs

PhD Defence

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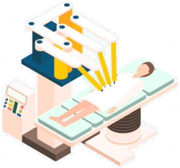
Critical software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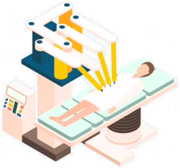


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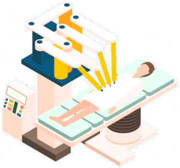


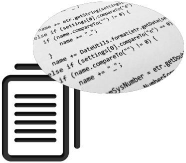

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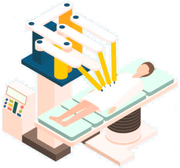

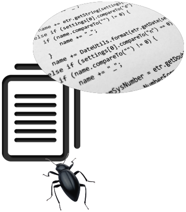

Proofs of correctness

Last year's Baccalauréat's philosophy exam:

BACCALAURÉAT GÉNÉRAL

Session 2016

PHILOSOPHIE

Série S

Sujet 2

Faut-il démontrer pour savoir ?

$\sim$ To know something, does one need a proof of it?

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$\sim$ To know something, does one need a proof of it?

Yes!

Imperative programs

List of instructions:

```
cmd1;
cmd2;
cmd3;
```

Relation between memory states:

Imperative programs

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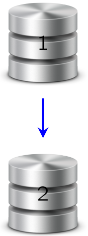

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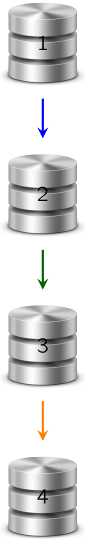

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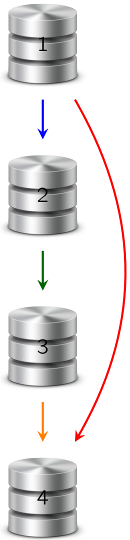

Relational program semantics

$x \leftarrow 1; ((y \leftarrow x) \oplus (y \leftarrow 0))$

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$\downarrow$

$a \cdot (b \cup c)$

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$\downarrow$

$(a \cdot b) \cup (a \cdot c)$

Relational program semantics

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$$\mathrm{Rel} \models a \cdot (b \cup c) = (a \cdot b) \cup (a \cdot c)$$

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Relation Algebra

Relational Operators

identity relation	1
empty relation	0
universal relation	$\top$
composition	$R \cdot S$
union	$R \cup S$
intersection	$R \cap S$
converse	$R^{\smile}$
reflexive transitive closure	$R^{\star}$
complement	R^{c}

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complement	R^{c}	$:= \{\langle x, y\rangle \mid \text{not } x\, R\, y\}$

Universal laws of relation algebra

Let O be a set, and R, S and T three binary relations over O.

$$\mathrm{Rel} \models 1 \cup R^{\star} \cdot S \subseteq (R \cup S)^{\star}$$

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Simple and boring

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$\hookrightarrow$ **No!** (Undecidable)

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Simple and boring : **could we have a computer do it?**

$\hookrightarrow$ Positive Relation Algebra

Outline

I. Introduction
- Motivation & Context
- Kleene Algebra
- Extensions

II. Kleene Allegory
- Terms and graphs
- Petri automata
- Comparing automata

III. Kleene Theorems
- Kleene theorem for simple Petri automata
- Kleene theorem for general Petri automata

IV. Other extensions

Outline

Regular expressions & languages

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Regular expressions

$$e, f \in \mathrm{Reg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^{\star}$$

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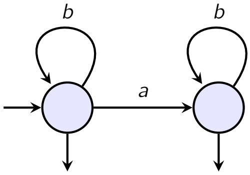
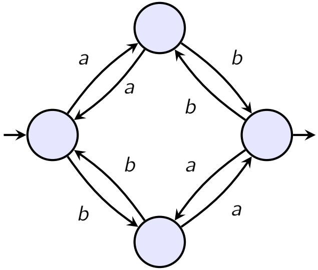
Examples

$$\mathcal{L}(a \cdot (b \cup c)) = \{ab, ac\} \qquad \mathcal{L}(a \cdot 0) = \emptyset$$

$$\mathcal{L}(a^\star) = \{\varepsilon, a, aa, \dots\} = \{a^n \mid n \in \mathbb{N}\}$$

Automata





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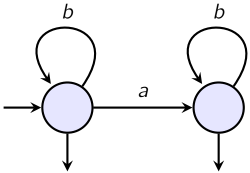
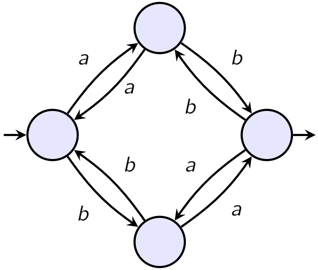
Theorem

Automata equivalence is decidable.





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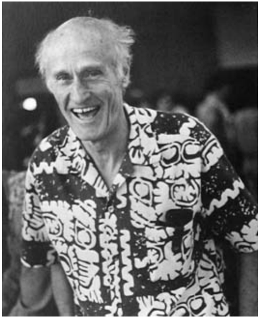
Kleene Theorem

A language is regular if and only if it is recognised by an automaton.



Stephen Cole Kleene

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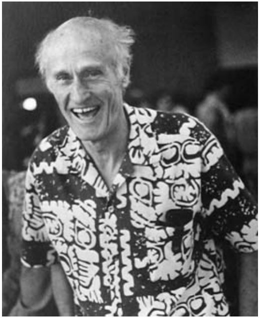
Corollary

There is an algorithm testing if two regular expressions e and f represent the same language.



Stephen Cole Kleene

Kleene Algebra

A Kleene algebra is an algebraic structure $\langle K, \cup, \cdot, ^\star, 0, 1\rangle$ such that

- $\langle K, \cup, \cdot, 0, 1\rangle$ is an idempotent semiring.
- The star satisfies the following laws:

$$1 \cup a \cdot a^\star \leqslant a^\star$$
$$b \cup a \cdot x \leqslant x \Rightarrow a^\star \cdot b \leqslant x$$

(+ symmetric)

John Horton Conway

Examples:

Algebras of languages: the set of languages over some finite alphabet A.

Algebras of relations: the set of binary relations over some set O.

Equivalence of expressions

Everything is equivalent

$$\forall O, \forall a, b, c, \ldots \in \mathcal{P}(O \times O), e = f?$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.

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- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.
- $\mathrm{Lang} \models e = f$: universal law of language Kleene algebra.

$$\mathrm{Rel} \models e = f \Longleftrightarrow \mathcal{L}\left(e\right) = \mathcal{L}\left(f\right) \Longleftrightarrow \mathrm{Lang} \models e = f$$

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- $\mathrm{KA} \vdash e = f$: universal law of Kleene algebra. (i.e. logical consequence of the axioms of KA).

$$\mathrm{KA} \vdash e = f$$

$$\Updownarrow$$

$$\mathrm{Rel} \models e = f \iff \mathcal{L}(e) = \mathcal{L}(f) \iff \mathrm{Lang} \models e = f$$

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decidable

Extensions of Kleene Algebra

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identity relation	:	1	✓
empty relation	:	0	✓
universal relation	:	$\top$	
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union	:	$R \cup S$	✓
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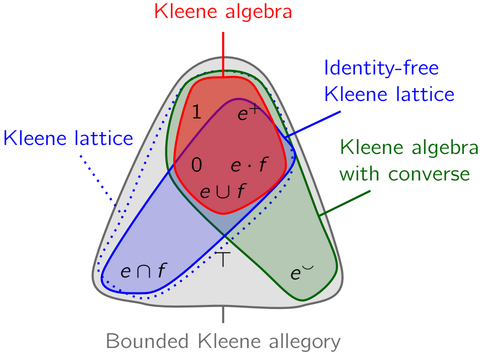

Equivalence of expressions

Regular expressions e, f.

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decidable

Equivalence of expressions

Richer expressions e, f.

$$\mathrm{KA} \vdash e = f$$

$$\Updownarrow$$

$$\mathrm{Rel} \models e = f \iff {}^{\blacktriangleleft}\mathcal{G}(e) = {}^{\blacktriangleleft}\mathcal{G}(f) \iff \mathrm{Lang} \models e = f$$

decidable ?

Outline

Kleene Allegories

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{\star} \mid e^{\smile} \mid \top$$

Kleene Allegories

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{\star} \mid e^{\smile} \mid \top$$

$$\mathrm{Rel} \models e = f \overset{?}{\not\Leftrightarrow} \mathcal{L}(e) = \mathcal{L}(f)$$

Kleene Allegories

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{\star} \mid e^{\smile} \mid \top$$

$$\mathrm{Rel} \models e = f \not\Leftarrow \mathcal{L}(e) = \mathcal{L}(f) \qquad \mathrm{Rel} \models e = f \not\Rightarrow \mathcal{L}(e) = \mathcal{L}(f)$$

Counterexample

$\mathcal{L}(a \cap b) = \mathcal{L}(0)_{=\emptyset}$	$\mathcal{L}(a) = \mathcal{L}(a^{\smile})_{=\{a\}}$	$\mathcal{L}(\top a \top b \top) \neq \mathcal{L}(\top b \top a \top)$
$\mathrm{Rel} \not\models a \cap b = 0$	$\mathrm{Rel} \not\models a = a^{\smile}$	$\mathrm{Rel} \models \top a \top b \top = \top b \top a \top$

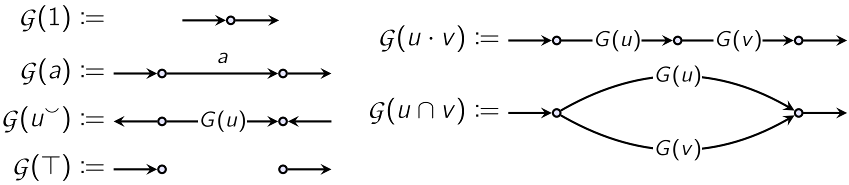
A different approach is needed.

Graphs/Ground terms

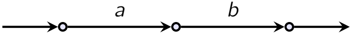
$$u, v \in \mathsf{W}_{\Sigma} ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$

Graphs/Ground terms

$$u, v \in \mathsf{W}_\Sigma ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$



Examples

$\mathcal{G}(a \cdot b)$: 

Examples

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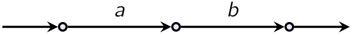
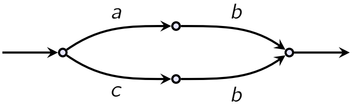


$\mathcal{G}((a \cdot b) \cap (c \cdot b))$:

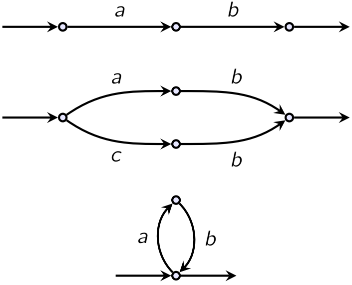


Examples

$\mathcal{G}(a \cdot b)$:

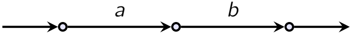
$\mathcal{G}((a \cdot b) \cap (c \cdot b))$:

$\mathcal{G}((a \cdot b) \cap 1)$:

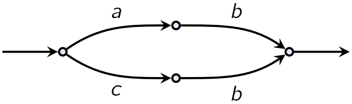


Examples

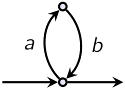
$\mathcal{G}\left(a \cdot b\right)$:



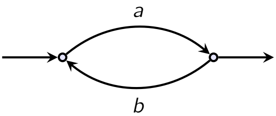
$\mathcal{G}\left((a \cdot b) \cap (c \cdot b)\right)$:



$\mathcal{G}\left((a \cdot b) \cap 1\right)$:

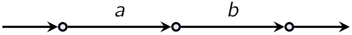


$\mathcal{G}\left(a \cap b^{\smile}\right)$:

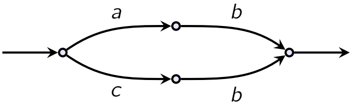


Examples

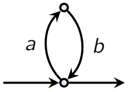
$\mathcal{G}\,(a \cdot b)$:



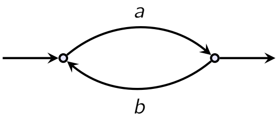
$\mathcal{G}\,((a \cdot b) \cap (c \cdot b))$:



$\mathcal{G}\,((a \cdot b) \cap 1)$:



$\mathcal{G}\,(a \cap b^{\smile})$:

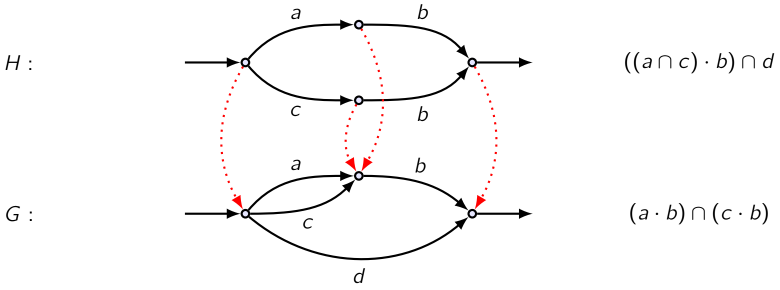


$\mathcal{G}\,(\top a \top b \top)$:

Preorder

Preorder on graphs

$G \blacktriangleleft H$ if there exists a graph morphism from H to G.



Characterization theorem

$$u, v \in \mathsf{W}_\Sigma ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$

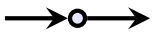
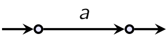
Theorem

$$\mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v)$$

- Freyd & Scedrov, **Categories, Allegories**, *1990* .
- Andréka & Bredikhin, **The equational theory of union-free algebras of relations**, *1995*

Graph languages

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^\star \mid e^\smile \mid \top$$

$\mathcal{G}(1) := \Big\{$  $\Big\}$ $\quad$ $\mathcal{G}(a) := \Big\{$  $\Big\}$ $\quad$ $\mathcal{G}(\top) := \Big\{$  $\Big\}$

$$\mathcal{G}(e^\smile) := \left\{ G^\smile \mid G \in \mathcal{G}(e) \right\}$$

$$\mathcal{G}(e \cdot f) := \{ G \cdot G' \mid G \in \mathcal{G}(e) \text{ and } G' \in \mathcal{G}(f) \}$$

$$\mathcal{G}(e \cap f) := \{ G \cap G' \mid G \in \mathcal{G}(e) \text{ and } G' \in \mathcal{G}(f) \}$$

$$\mathcal{G}(0) := \emptyset \qquad \mathcal{G}(e \cup f) := \mathcal{G}(e) \cup \mathcal{G}(f) \qquad \mathcal{G}(e^\star) := \bigcup_{n \in \mathbb{N}} \mathcal{G}(e)^n .$$

Characterization theorem

$^{\blacktriangleleft}S$ is the downwards closure of S with respect to $\blacktriangleleft$.
$^{\blacktriangleleft}S := \{G \mid \exists H \in S : G \blacktriangleleft H\}$.

Theorem

$e, f \in \mathrm{AReg}\langle\Sigma\rangle$,

$$\mathrm{Rel} \models e \subseteq f \Leftrightarrow {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f)$$

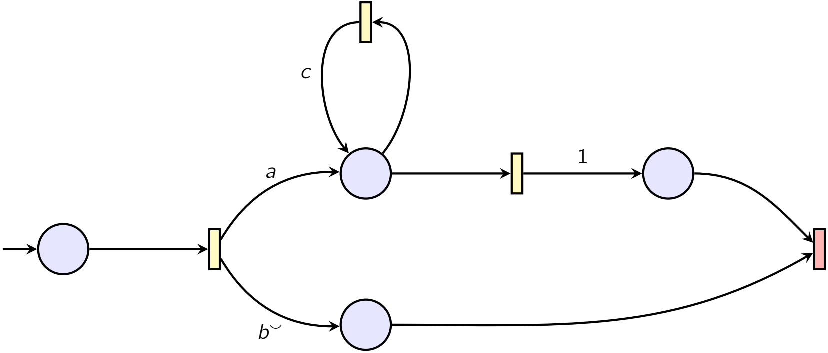
B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Follows easily from:

Andréka, Mikulás & Németi, **The equational theory of Kleene lattices**, *TCS'11*

Outline

Petri automata



Set of labels: $\Sigma' := \Sigma \cup \left\{ a^{\smile} \mid a \in \Sigma \right\} \cup \{1, \top\}$.

Language generated vs. Language recognised

Let $\mathcal{A}$ be a finite state word automaton.

The language of $\mathcal{A}$ is:

Language generated vs. Language recognised

Let $\mathcal{A}$ be a finite state word automaton.

The language of $\mathcal{A}$ is:

- the set of words **accepted** by $\mathcal{A}$; **recognised**

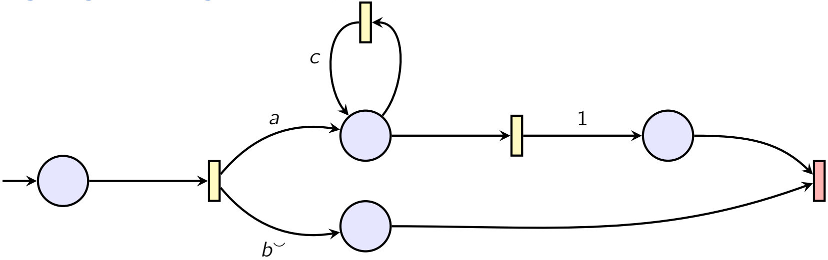
Language generated vs. Language recognised

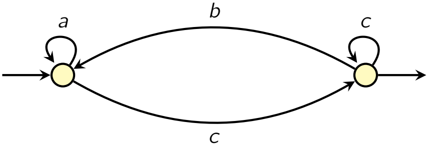
Let $\mathcal{A}$ be a finite state word automaton.

The language of $\mathcal{A}$ is:

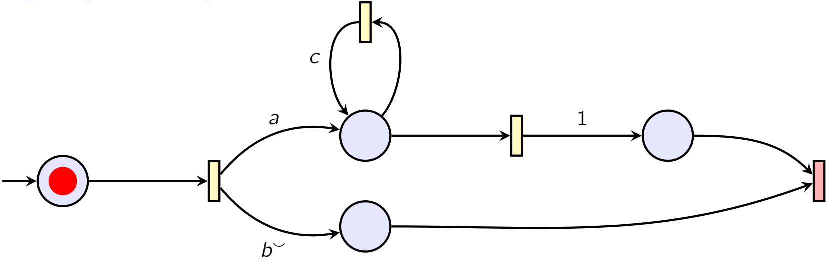
- the set of words **accepted** by $\mathcal{A}$; **recognised**
- the set of words labelling **accepting paths** in $\mathcal{A}$. **generated**

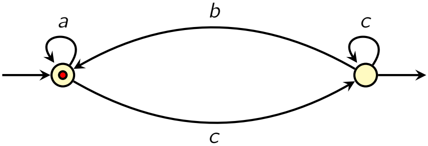
Language recognised by an automaton



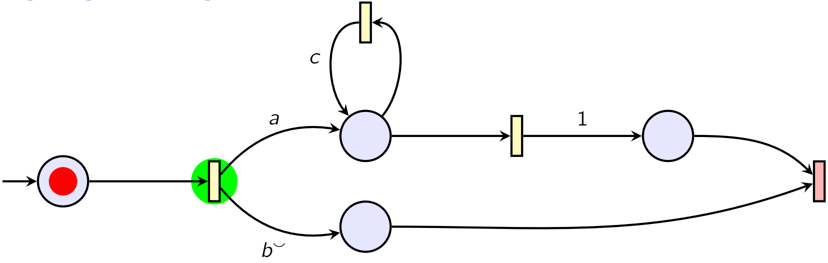


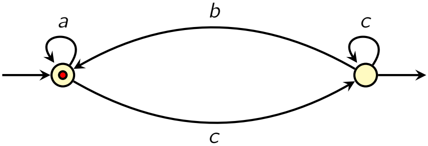
Language recognised by an automaton





Language recognised by an automaton





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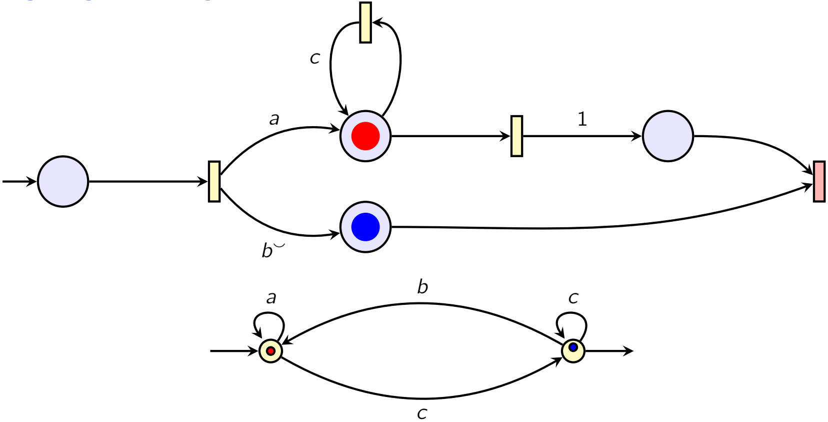
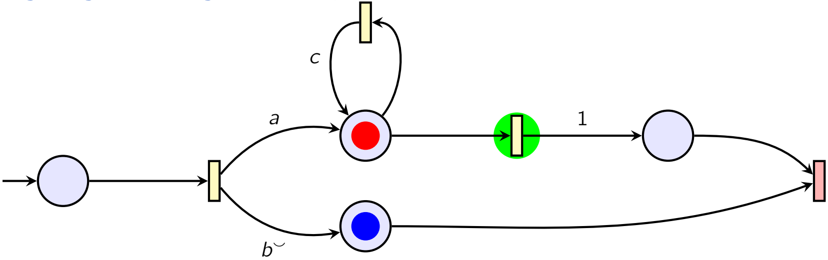
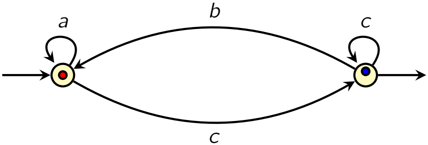
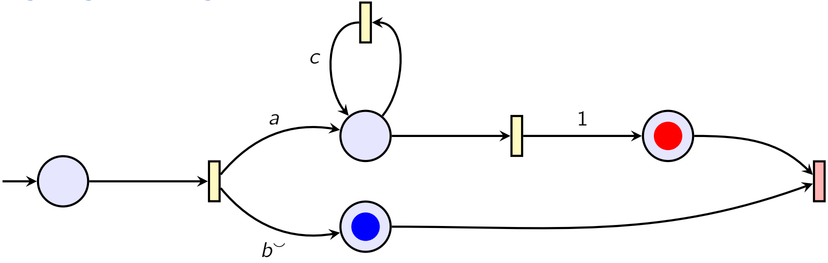


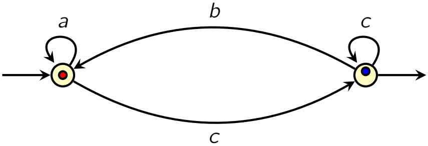
Language recognised by an automaton





Language recognised by an automaton





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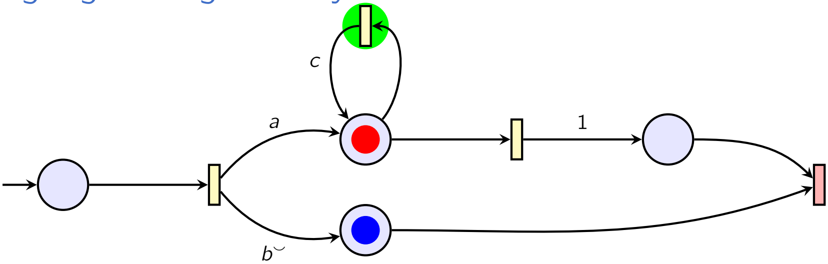
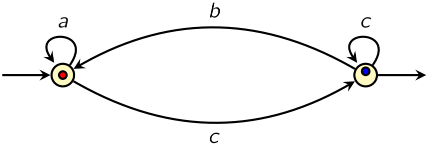
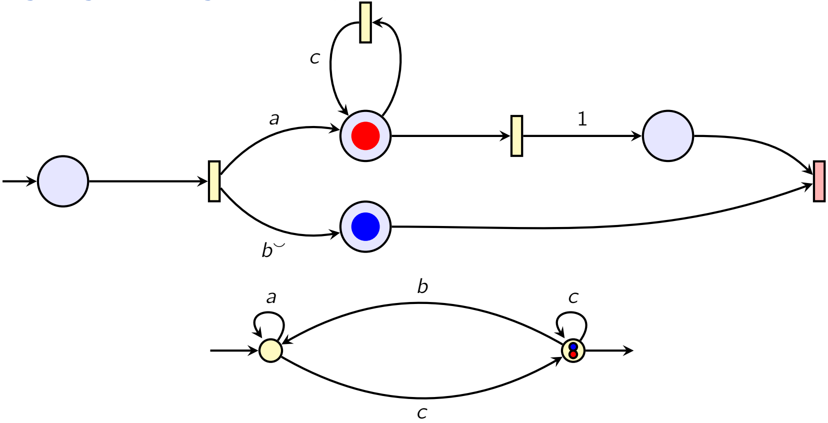
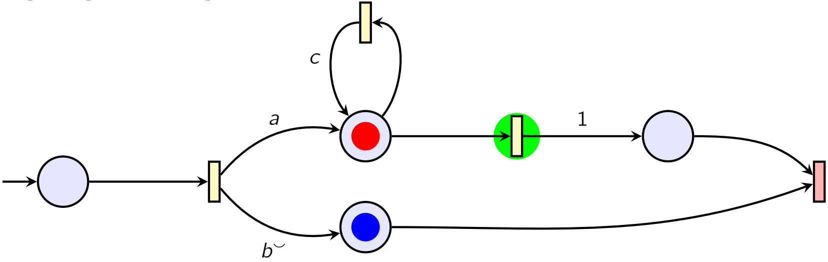


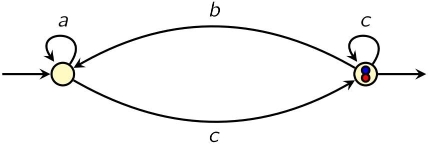


Language recognised by an automaton



Language recognised by an automaton





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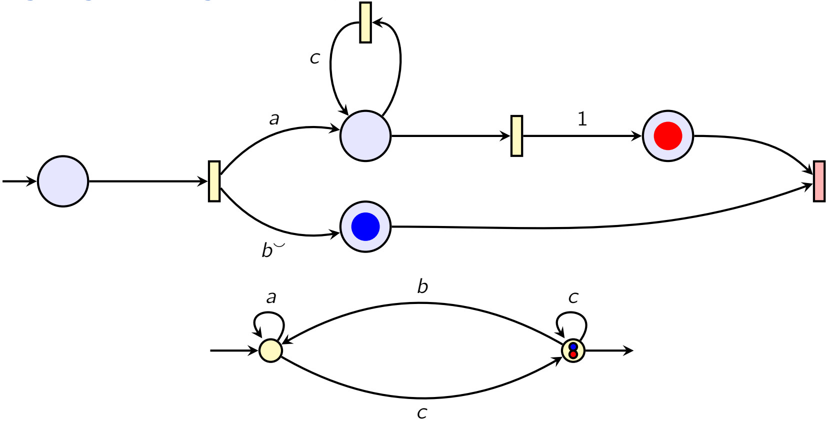

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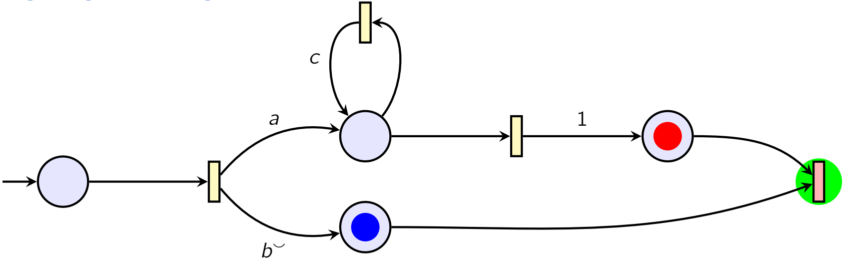

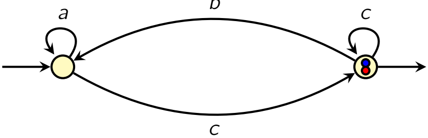

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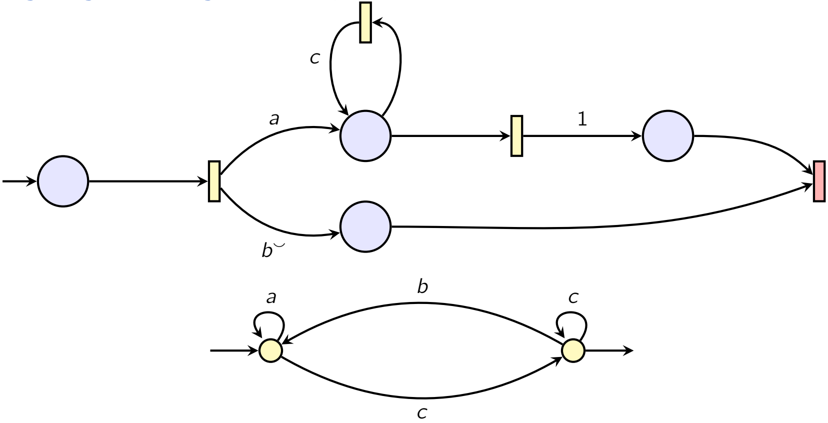
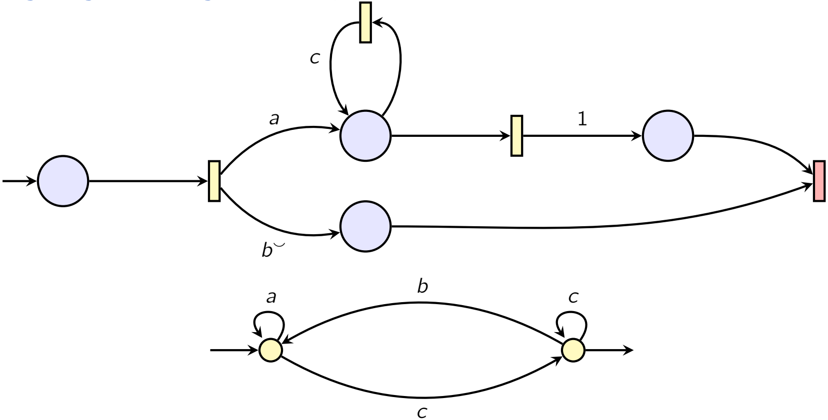


Success!

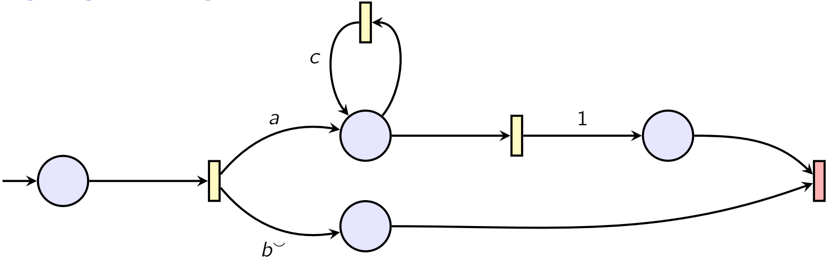
Language recognised by an automaton

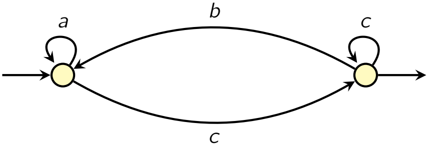


Language of $\mathcal{A}$

$$\mathcal{L}(\mathcal{A}) = \{G \mid \mathcal{A} \text{ accepts } G\}$$

Language recognised by an automaton





Language of $\mathcal{A}$

$$\mathcal{L}(\mathcal{A}) = \{G \mid \mathcal{A} \text{ accepts } G\}$$

Here: $\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\mathcal{G}\left((a \cdot c^{\star}) \cap b^{\smile}\right)$.

Petri automata for Kleene allegories

Correctness

For any $e \in \text{AReg}\langle\Sigma\rangle$,

$$e$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathrm{AReg}\langle\Sigma\rangle$,

$$\mathcal{A}(e)$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathrm{AReg}\langle\Sigma\rangle$,

$$\mathcal{L}\left(\mathcal{A}(e)\right)$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathrm{AReg}\langle\Sigma\rangle$,

$$\mathcal{L}\left(\mathcal{A}(e)\right) = {}^{\blacktriangleleft}\mathcal{G}\left(e\right).$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Petri automata for Kleene allegories

Correctness

For any $e \in \mathrm{AReg}\langle\Sigma\rangle$,

$$\mathcal{L}(\mathcal{A}(e)) = {}^{\blacktriangleleft}\mathcal{G}(e).$$

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

So far:

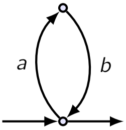
$e, f \in \mathrm{AReg}\langle\Sigma\rangle$

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f) \;\Leftrightarrow\; \mathcal{L}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)).$$

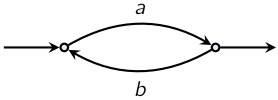
Outline

Restriction: identity-free Kleene lattice terms

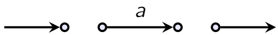
$\mathcal{G}((a \cdot b) \cap 1)$:



$\mathcal{G}\left(a \cap b^{\smile}\right)$:

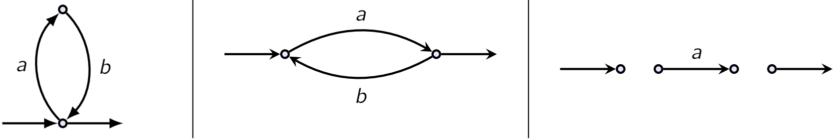


$\mathcal{G}(\top a \top)$:



Restriction: identity-free Kleene lattice terms

$\mathcal{G}((a \cdot b) \cap 1)$: $\mathcal{G}(a \cap b^{\smile})$: $\mathcal{G}(\top a \top)$:



Identity-free Kleene Lattice

$$u, v \in \mathrm{W}_\Sigma^- ::= 1 \mid a \mid u \cdot v \mid u \cap v \mid u^{\smile} \mid \top$$

$$e, f \in \mathrm{GReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{+} \mid e^{\smile} \mid \top$$

$\hookrightarrow$ Series-Parallel graphs & simple automata

Decision procedure

$e, f \in \mathsf{GReg}\langle\Sigma\rangle$

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f) \;\Leftrightarrow\; \mathcal{L}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)).$$

Problem:

How to compare two simple Petri automata?

Decision procedure

$e, f \in \mathsf{GReg}\langle\Sigma\rangle$

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; {}^{\blacktriangleleft}\mathcal{G}(e) \subseteq {}^{\blacktriangleleft}\mathcal{G}(f) \;\Leftrightarrow\; \mathcal{L}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)).$$

Problem:

How to compare two simple Petri automata?

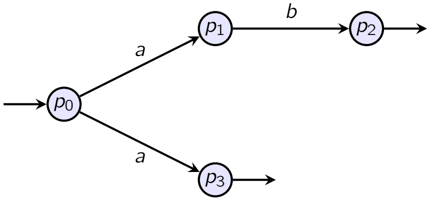
$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$ if and only if there is a simulation relation

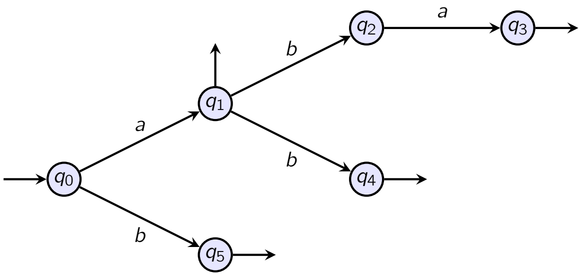
$$\preccurlyeq \subseteq \mathcal{P}\left(P_1\right) \times \mathcal{P}\left(P_2 \rightharpoonup P_1\right)$$

between the configurations of $\mathcal{A}_1$ and the partial maps from the places of $\mathcal{A}_2$ to the places of $\mathcal{A}_1$.

Simulations - non-deterministic finite automata

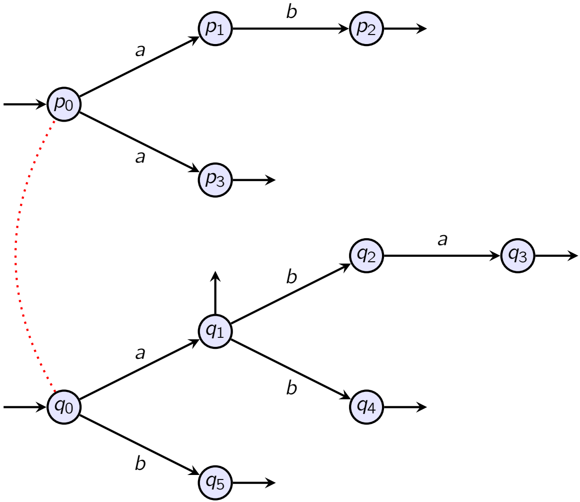
$$\preccurlyeq \subseteq Q_1 \times \mathcal{P}(Q_2)$$





Simulations - non-deterministic finite automata

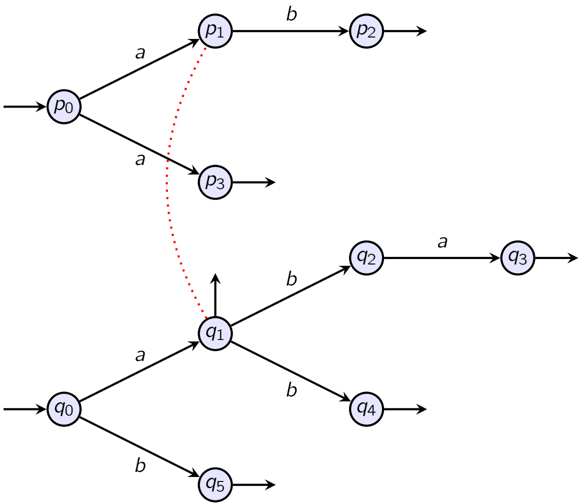
$$\preccurlyeq \subseteq Q_1 \times \mathcal{P}(Q_2)$$



$p_0 \preccurlyeq \{ \quad q_0 \quad \}$

Simulations - non-deterministic finite automata

$$\preccurlyeq \subseteq Q_1 \times \mathcal{P}(Q_2)$$



$p_0 \preccurlyeq \{ \; q_0 \; \}$
$p_1 \preccurlyeq \{ \; q_1 \; \}$

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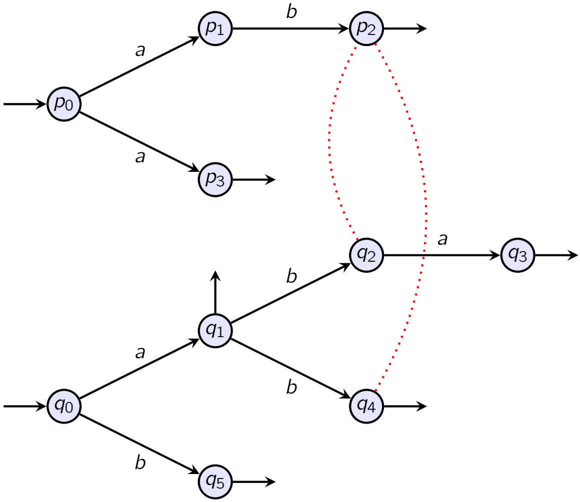

$p_0 \preccurlyeq \{ q_0 \}$
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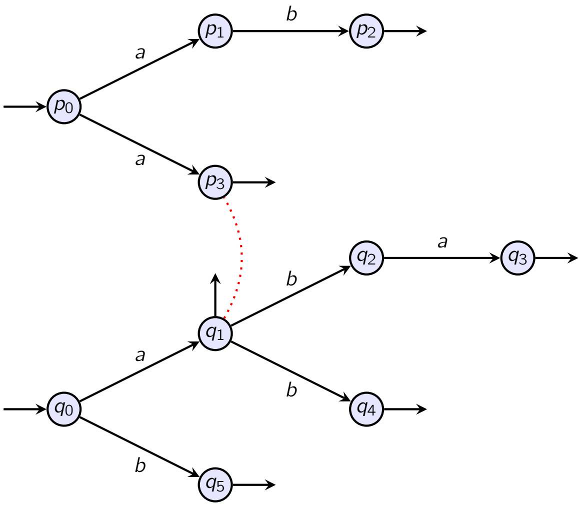

$$\begin{aligned} p_0 &\preccurlyeq \{q_0\} \\ p_1 &\preccurlyeq \{q_1\} \\ p_2 &\preccurlyeq \{q_2, q_4\} \\ p_3 &\preccurlyeq \{q_1\} \end{aligned}$$

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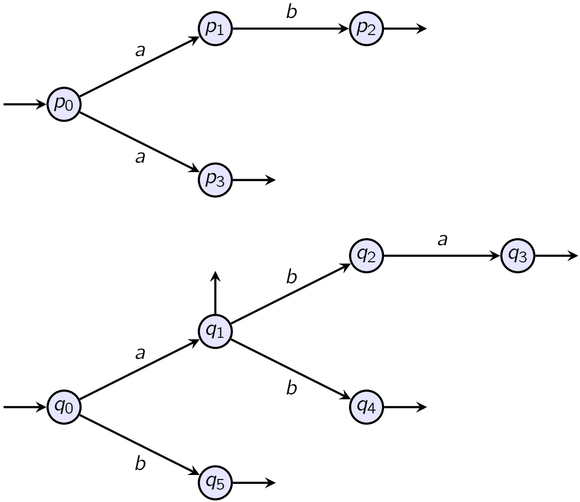

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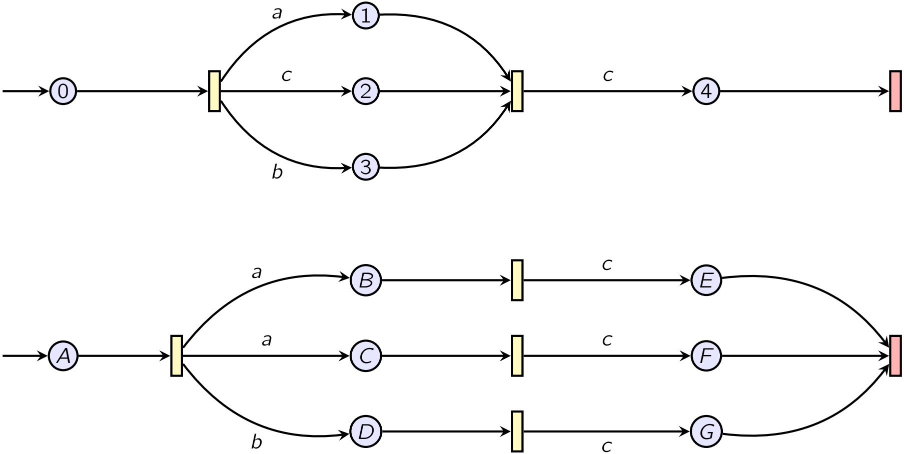
+ condition for accepting states

Simulation - Petri automata

Simulation relation

$$\preccurlyeq \subseteq \mathcal{P}(P_1) \times \mathcal{P}(P_2 \rightharpoonup P_1)$$



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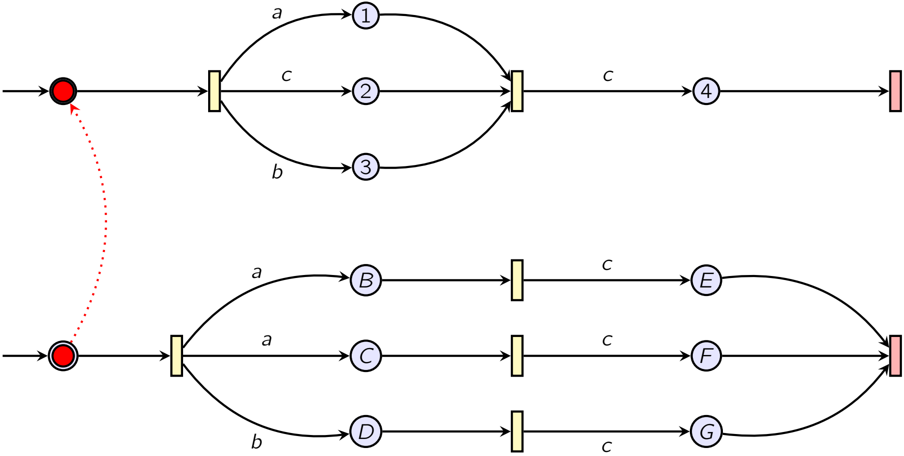

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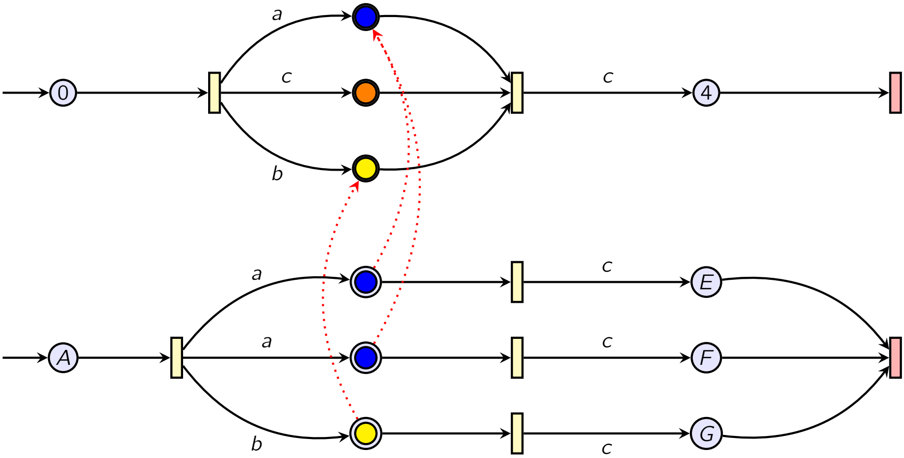

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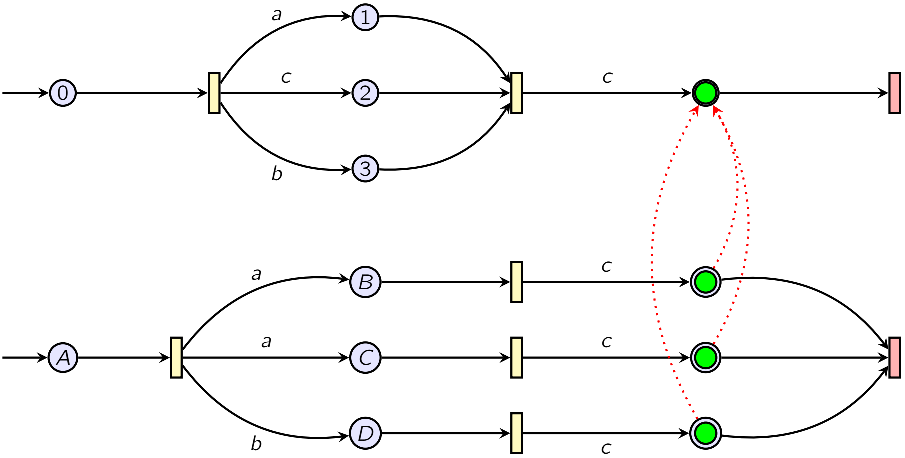

Comparing Petri automata

Theorem

Comparing simple Petri automata is **ExpSpace-complete**.

Proof.

ExpSpace easy: testing for the existence of a simulation can be done in exponential space;

ExpSpace hard: reduction from the universality problem for regular expressions with squaring.

□

B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

Meyer & Stockmeyer, **The equivalence problem for regular expressions with squaring requires exponential space**, *SWAT.'72*

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Corollary

Relational equivalence for Identity-free Kleene lattices is **decidable**.

Outline

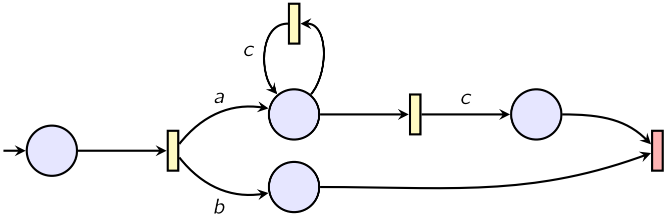
I. Introduction
- Motivation & Context
- Kleene Algebra
- Extensions

II. Kleene Allegory
- Terms and graphs
- Petri automata
- Comparing automata

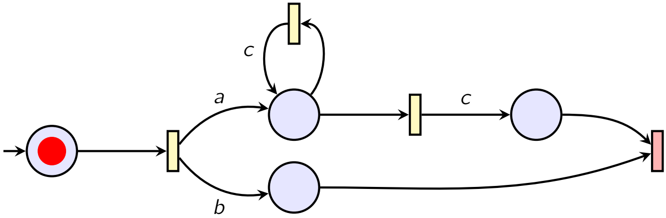
III. Kleene Theorems
- Kleene theorem for simple Petri automata
- Kleene theorem for general Petri automata

IV. Other extensions

Trace language of an automaton



Trace language of an automaton



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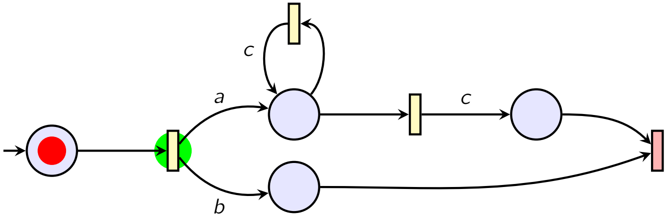

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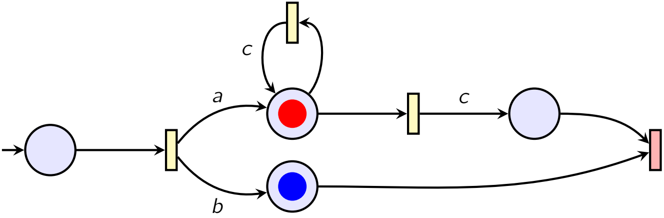

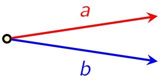

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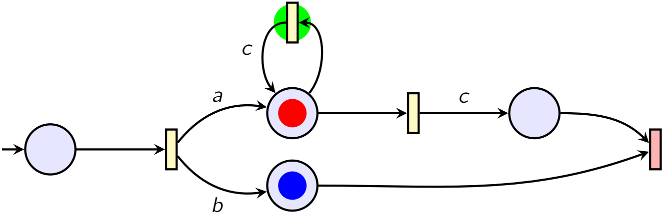

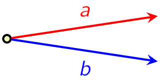

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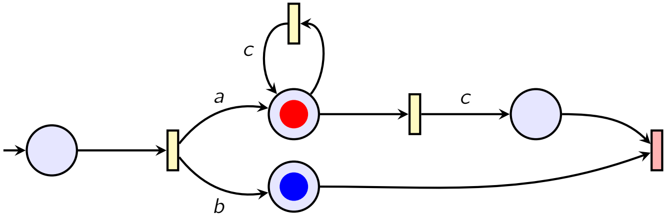

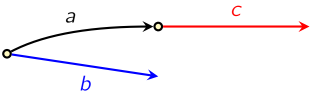

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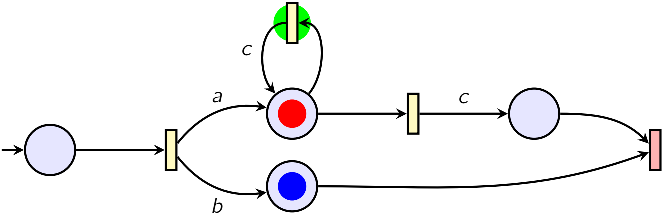

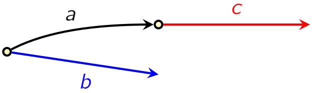

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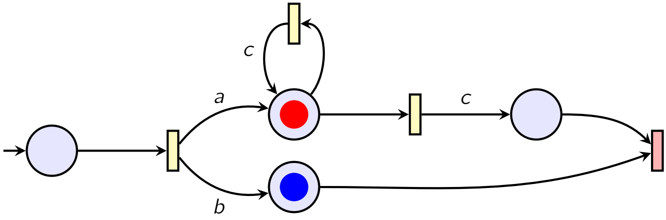

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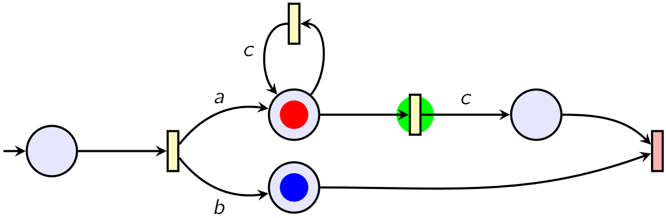

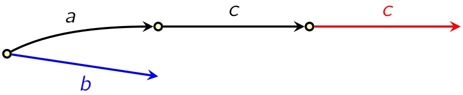

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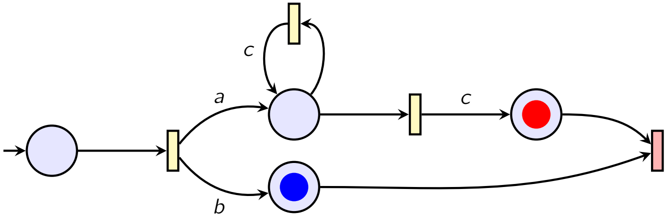

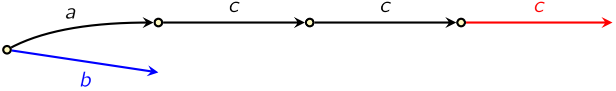

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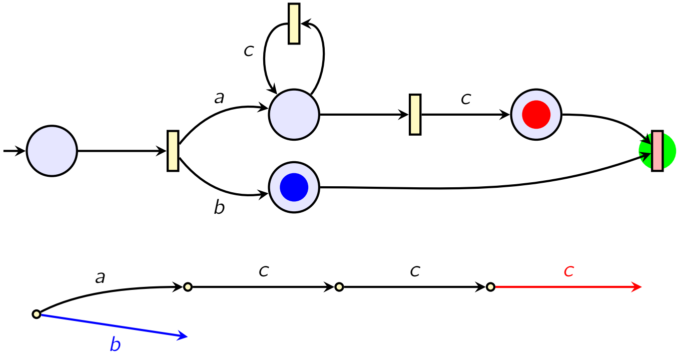

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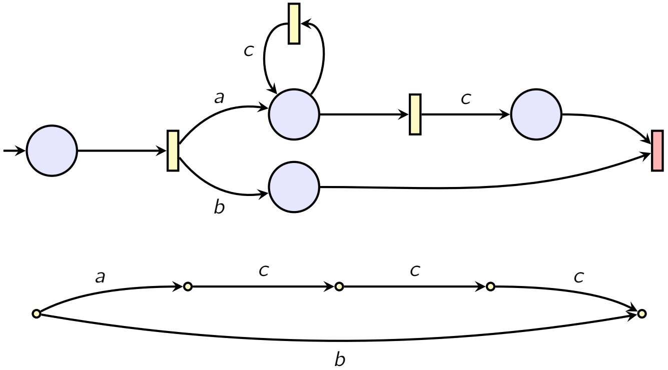

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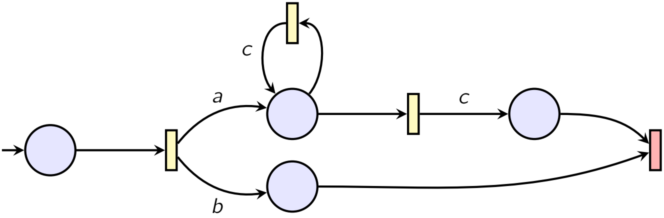
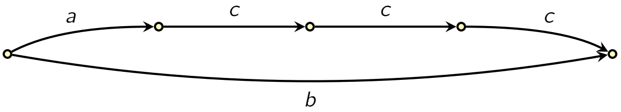
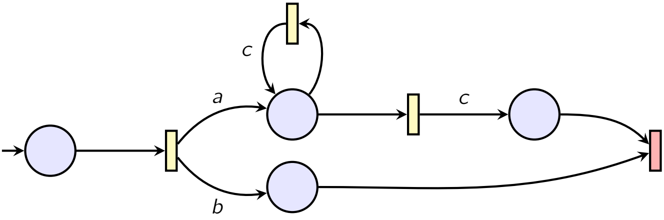


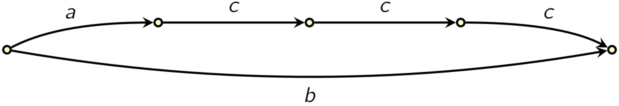


Trace language of $\mathcal{A}$

$$\mathcal{T}r(\mathcal{A}) = \{G \mid G \text{ is the trace of an accepting run in } \mathcal{A}\}$$

Trace language of an automaton





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Here: $\mathcal{T}r(\mathcal{A}) = \mathcal{G}((a \cdot c^+) \cap b)$.

Traces, languages and expressions

Lemma

If $\mathcal{A}$ is only labelled with Σ,

$$\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\mathcal{T}r(\mathcal{A}).$$

Fact

$e \in \mathsf{GReg}\langle\Sigma\rangle$,

$$\mathcal{T}r(\mathcal{A}(e)) = \mathcal{G}(e).$$

Kleene Theorem

Definitions

A set of SP graphs $\mathcal{G}$ is:

- regular: if there is an expression e in $\mathrm{GReg}\langle\Sigma\rangle$ such that $\mathcal{G}(e) = \mathcal{G}$;
- recognisable: if there is a simple Petri automaton $\mathcal{A}$ such that $\mathcal{T}r(\mathcal{A}) = \mathcal{G}$.

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Theorem

The class of regular sets of graphs coincides with the recognisable sets of graphs.

Kleene Theorem

Definitions

A set of SP graphs $\mathcal{G}$ is:

regular if there is an expression e in $\mathrm{GReg}\langle\Sigma\rangle$ such that $\mathcal{G}(e) = \mathcal{G}$;

recognisable if there is a simple Petri automaton $\mathcal{A}$ such that $\mathcal{Tr}(\mathcal{A}) = \mathcal{G}$.

Theorem

The class of regular sets of graphs coincides with the recognisable sets of graphs.

Proof.

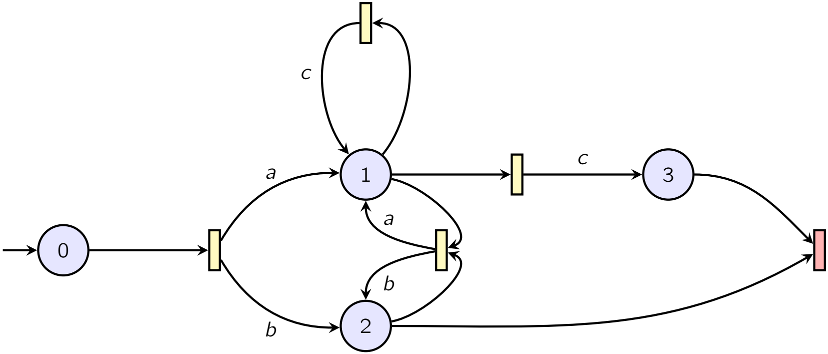
- **Expressions to automata**: inductive construction;

  B. & Pous, **Petri automata for Kleene Allegories**, *LICS'15*

- **Automata to expressions**: next slide.

□

From automata to expressions



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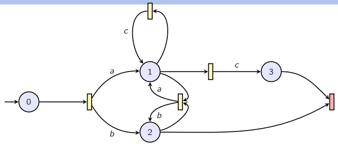

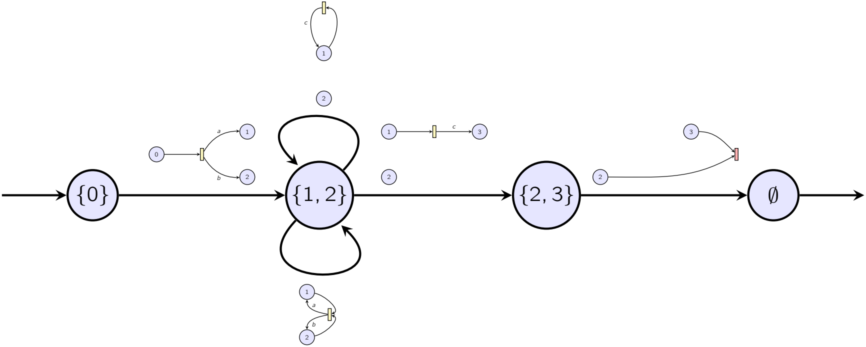

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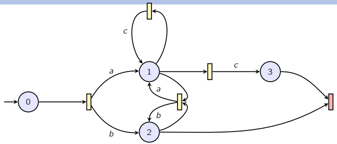

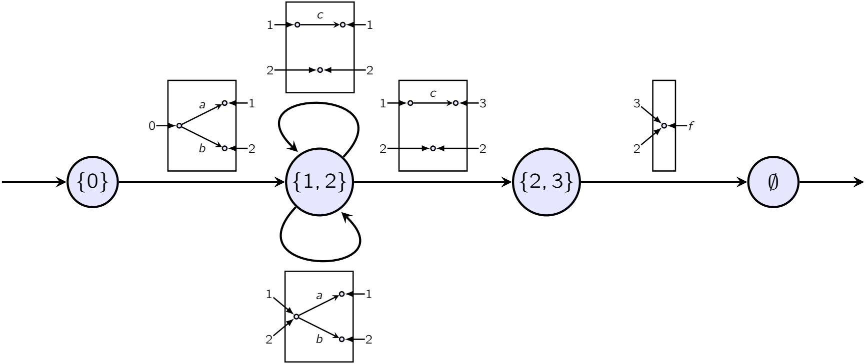

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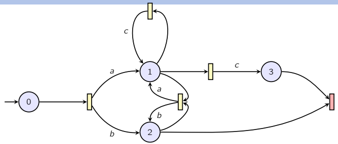

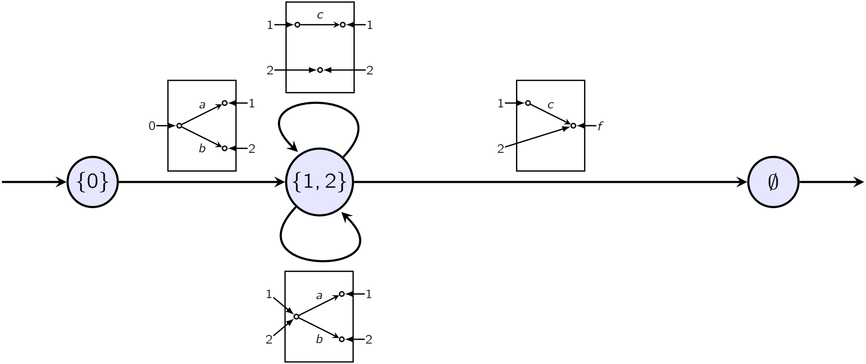

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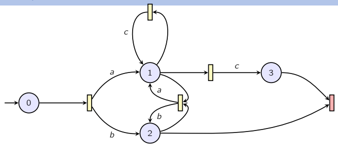

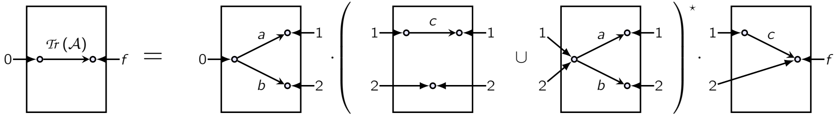

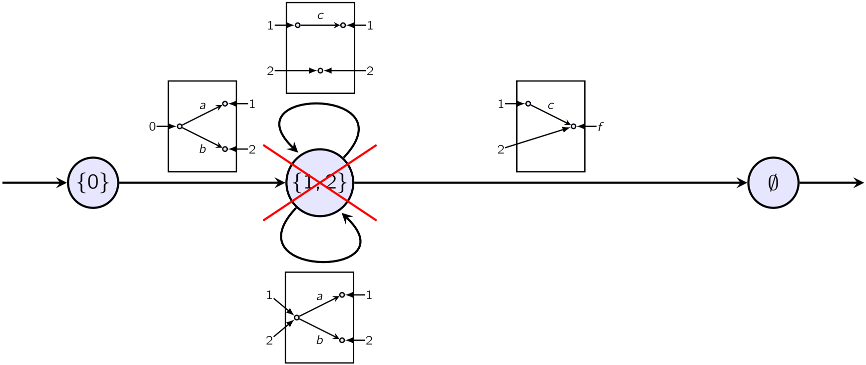

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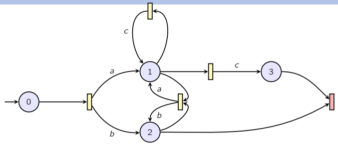

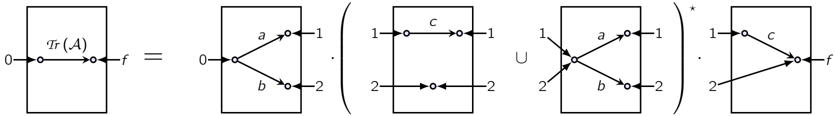

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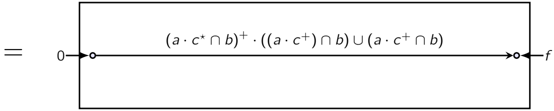
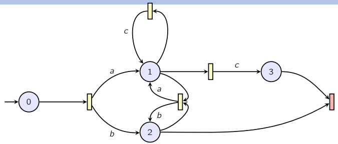
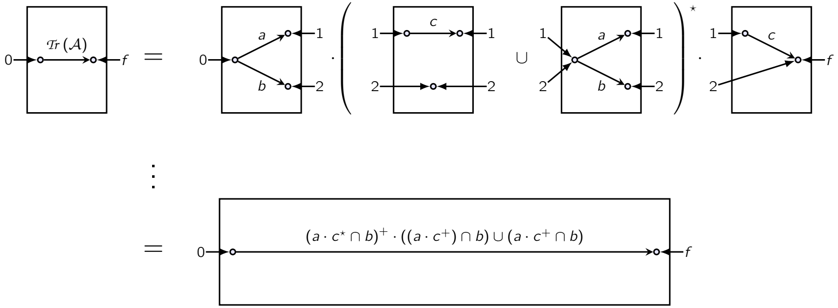


(With some effort...)

From automata to expressions





Result

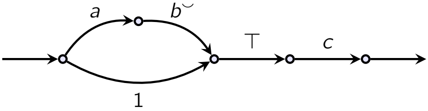
$$\mathcal{E}\,(\mathcal{A}) = (a \cdot c^\star \cap b)^+ \cdot \left(\left(a \cdot c^+\right) \cap b\right) \cup \left(a \cdot c^+ \cap b\right)$$

$$\mathcal{T}r\,(\mathcal{A}) = \mathcal{G}\,(\mathcal{E}\,(\mathcal{A}))$$

Outline

Relationship between $\mathcal{Tr}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$

Φ : graphs over $\Sigma' \to$ graphs over Σ.



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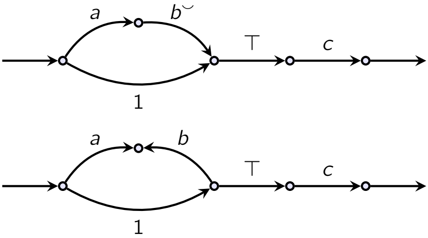

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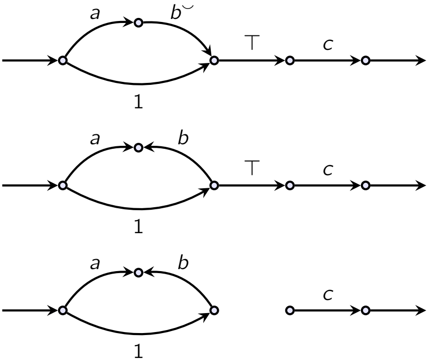

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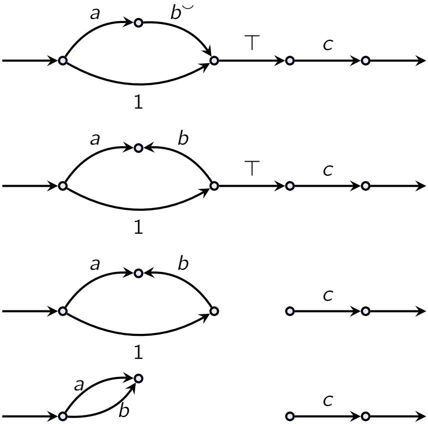

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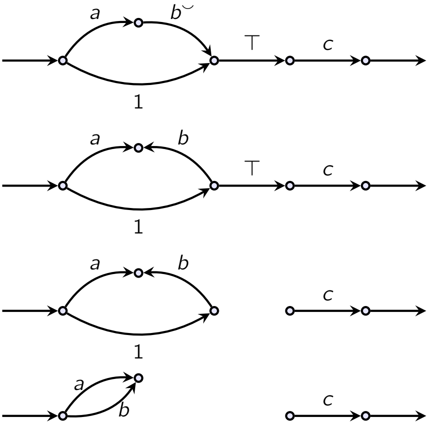

Lemma

$$\mathcal{L}(\mathcal{A}) = {}^{\blacktriangleleft}\Phi(\mathcal{Tr}(\mathcal{A})).$$

From $\mathrm{AReg}\langle\Sigma\rangle$ to $\mathrm{GReg}\langle\Sigma'\rangle$

$$e, f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{\star} \mid e^{\smile} \mid \top$$

$$e, f \in \mathrm{GReg}\langle\Sigma\rangle ::= 0 \mid a \mid e \cdot f \mid e \cap f \mid e \cup f \mid e^{+}$$

$$\Sigma' := \Sigma \cup \left\{ a^{\smile} \mid a \in \Sigma \right\} \cup \{1, \top\}.$$

$$e = (a \cdot b^{\star})^{\smile} \cup \top$$

From AReg$\langle\Sigma\rangle$ to GReg$\langle\Sigma'\rangle$

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$$\Sigma' := \Sigma \cup \left\{ a^\smile \mid a \in \Sigma \right\} \cup \{1, \top\}.$$

$$\begin{aligned} e &= (a \cdot b^\star)^\smile \cup \top \\ &\equiv \left(a \cdot (1 \cup b^+)\right)^\smile \cup \top \\ &\equiv \left(1 \cup \left(b^\smile\right)^+\right) \cdot a^\smile \cup \top \in \mathrm{GReg}\langle\Sigma'\rangle \\ &=: \varphi(e) \quad . \end{aligned}$$

From $\mathrm{AReg}\langle\Sigma\rangle$ to $\mathrm{GReg}\langle\Sigma'\rangle$

$$e,f \in \mathrm{AReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e\cdot f \mid e\cap f \mid e\cup f \mid e^\star \mid e^\smile \mid \top$$
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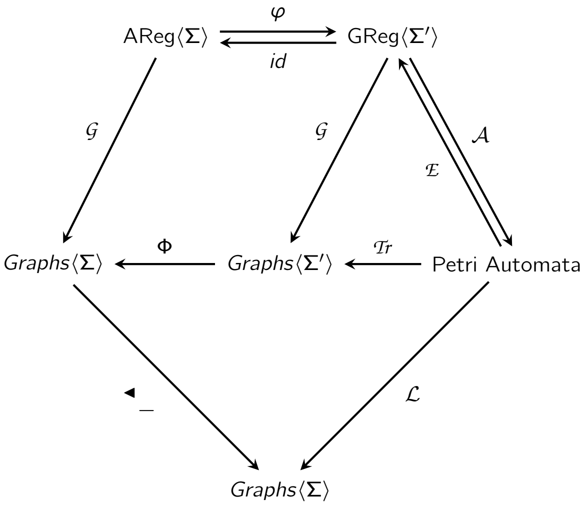
&=: \varphi(e) \quad .
\end{aligned}$$

$\varphi : \mathrm{AReg}\langle\Sigma\rangle \to \mathrm{GReg}\langle\Sigma'\rangle$

Lemma

$$\mathcal{G}_\Sigma(e) = \Phi\left(\mathcal{G}_{\Sigma'}(\varphi(e))\right).$$

Correspondences



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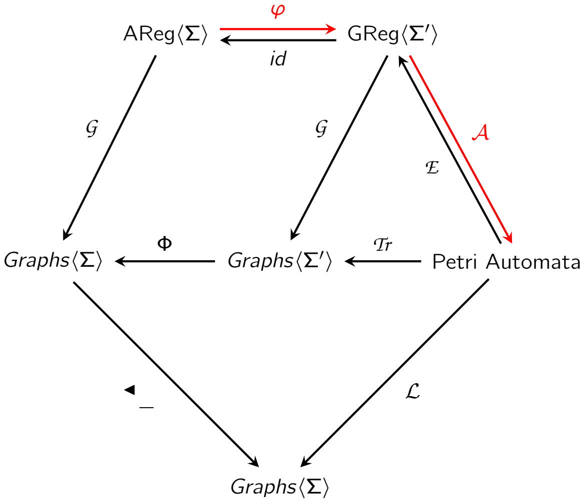

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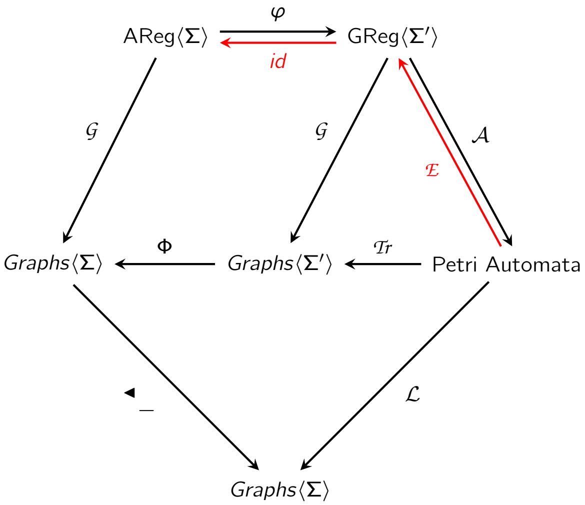

Kleene Theorem

Let $\mathcal{G}$ be a set of graphs labelled with Σ.

Theorem

$\mathcal{G}$ is the language recognised by a Petri automaton if and only if it is the closed graph language of an expression.

Corollary

The equational theory of bounded Kleene allegories is decidable if and only if the comparison of general Petri automata is decidable.

Overview

Results

- Reduction of relational equivalence to equality of closed graph languages.

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This decision procedure was implemented in OCaml, and is available online: `http://perso.ens-lyon.fr/paul.brunet/rklm`.

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Outline

Kleene Algebra with converse

$$e, f ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^{\star} \mid e^{\smile}.$$

Theorem

Deciding the equational theory of Kleene Algebra with converse is **PSpace-complete**.

- Bloom, Ésik & Stefanescu, **Notes on equational theories of relations**, *1995*
- Ésik & Bernátsky, **Equational properties of Kleene algebras of relations with conversion**, *TCS'95*
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Nominal Kleene Algebra

$$t \leftarrow x \qquad ; \qquad x \leftarrow y \qquad ; \qquad y \leftarrow t$$

B. & Pous, **A formal exploration of Nominal Kleene Algebra**, *MFCS'16*

Nominal Kleene Algebra

$t \leftarrow x \quad ; \quad x \leftarrow y \quad ; \quad y \leftarrow t$

$tx \qquad xy \qquad yt$

t	v_1		t	v_2
x	v_2		x	v_2
y	v_3		y	v_3

t	v_1		t	v_1
x	v_2		x	v_3
y	v_3		y	v_3

t	v_1		t	v_1
x	v_2		x	v_2
y	v_3		y	v_1

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$$t \leftarrow x \quad ; \quad x \leftarrow y \quad ; \quad y \leftarrow t$$

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Nominal Kleene Algebra

$\texttt{local var}\ t\ \texttt{in}\{\quad t \leftarrow x \quad ; \quad x \leftarrow y \quad ; \quad y \leftarrow t \quad \}$

$tx \quad \circ \quad xy \quad \circ \quad yt$

t	v_1
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Nominal Kleene Algebra

`local var` t `in`{	$t \leftarrow x$	;	$x \leftarrow y$	;	$y \leftarrow t$	}
$(\nu t.$	tx	$\circ$	xy	$\circ$	yt	$)$

x	v_2
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B. & Pous, **A formal exploration of Nominal Kleene Algebra**, *MFCS'16*

Developed using the proof assistant Coq ($\sim$ 9000 lines of proof script).

`http://perso.ens-lyon.fr/paul.brunet/nka`

That's all folks!

Thank you!

See more at:
http://perso.ens-lyon.fr/paul.brunet

Outline

I. Introduction
- Motivation & Context
- Kleene Algebra
- Extensions

II. Kleene Allegory
- Terms and graphs
- Petri automata
- Comparing automata

III. Kleene Theorems
- Kleene theorem for simple Petri automata
- Kleene theorem for general Petri automata

IV. Other extensions