The Equational Theory of Algebras of Languages

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Introduction

$\forall\Sigma,\ \forall L, M, N \subseteq \Sigma^\star,$

$$\left(L^\smile \cap M\right) \cdot \left(1 \cap \left(N^\star \cdot L^\smile\right)\right) = (1 \cap L) \cdot \left(L \cap M^\smile\right)^\smile$$

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- Decidability

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- Decidability
- Complexity

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- Decidability
- Complexity
- Axiomatisation

Language Algebra

Language Operators

unit language	1
empty language	0
composition	$L \cdot M$
union	$L \cup M$
intersection	$L \cap M$
mirror image	$L^{\smile}$
Kleene star	$L^{\star}$

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Kleene star	$L^{\star}$	$:= 1 \cup L \cup LL \cup LLLL \cup \dots$

Kleene Algebra

Where everything is nice

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Regular expressions

$$e, f \in \mathrm{Reg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^{\star} \mid e \cap f \mid e^{\smile}$$

- $\mathrm{Rel} \models e = f$: universal law of relational Kleene algebra.
- $\mathrm{Lang} \models e = f$: universal law of language Kleene algebra.
- $\mathrm{KA} \vdash e = f$: universal law of Kleene algebra.
  (i.e. logical consequence of the axioms of KA).

$$\mathrm{KA} \vdash e = f$$

$$\Updownarrow$$

$$\mathrm{Rel} \models e = f \iff \mathcal{L}(e) = \mathcal{L}(f) \iff \mathrm{Lang} \models e = f$$

decidable

Intersection & Mirror

$$e, f \in \mathsf{IMReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^{\star} \mid e \cap f \mid e^{\smile}$$

Intersection & Mirror

Language Algebra is not Relation Algebra

$$e, f \in \mathsf{IMReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^\star \mid e \cap f \mid e^\smile$$

$$\mathrm{Lang} \not\models a \leq a \cdot a^\smile \cdot a \qquad \text{but} \qquad \mathrm{Rel} \models a \leq a \cdot a^\smile \cdot a$$

$$\mathrm{Lang} \models (1 \cap a) \cdot b = b \cdot (1 \cap a) \qquad \text{but} \qquad \mathrm{Rel} \not\models (1 \cap a) \cdot b = b \cdot (1 \cap a)$$

Intersection & Mirror

$$e, f \in \mathsf{IMReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^{\star} \mid e \cap f \mid e^{\smile}$$

$$\mathrm{Lang} \models e = f \overset{?}{\Leftrightarrow} \mathcal{L}(e) = \mathcal{L}(f)$$

Intersection & Mirror

Regular languages are not enough

$$e, f \in \mathsf{IMReg}\langle\Sigma\rangle ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^\star \mid e \cap f \mid e^\smile$$

$$\mathrm{Lang} \models e = f \;\not\Leftarrow\; \mathcal{L}(e) = \mathcal{L}(f) \qquad \mathrm{Lang} \models e = f \;\Rightarrow\; \mathcal{L}(e) = \mathcal{L}(f)$$

Counterexample

$\mathcal{L}(a \cap b) = \mathcal{L}(0)$	$\mathcal{L}(a) = \mathcal{L}(a^\smile)$
$\mathrm{Lang} \not\models a \cap b = 0$	$\mathrm{Lang} \not\models a = a^\smile$

Outline

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Monoidal meet-semilattice

$$e, f \in \mathbb{T}_\Sigma ::= 1 \mid a \mid e \cdot f \mid e \cap f$$

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Proof. Let $\sigma : \{a, b\} \to \mathcal{P}(X^\star)$.

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- If $\epsilon \in \sigma(a)$:

$$\begin{aligned} \widehat{\sigma}((1 \cap a) \cdot b) &= \\ \widehat{\sigma}(b \cdot (1 \cap a)) &= \end{aligned}$$

- If $\epsilon \notin \sigma(a)$:

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□

Weak terms

$$e, f \in \mathbb{SP}_\Sigma ::= a \mid e \cdot f \mid e \cap f$$

Definition

A weak term is a pair $\langle t, A\rangle$ where $A \subseteq \Sigma$ and either $t = 1$ or $t \in \mathbb{SP}_\Sigma$. The set of weak terms is $\mathbb{W}_\Sigma$.

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Given a map $\sigma : \Sigma \to \mathcal{P}(X^\star)$, we define:

$$\tilde{\sigma}\langle t, A \rangle = \begin{cases} \emptyset & \textit{if } \exists a \in A : \epsilon \notin \sigma(a) \\ \widehat{\sigma}(t) & \textit{otherwise} \end{cases}$$

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$$\langle 1, A\rangle \bullet \langle 1, B\rangle \coloneqq \langle 1, A \cup B\rangle$$
$$\langle u, A\rangle \bullet \langle 1, B\rangle \coloneqq \langle u, A \cup B\rangle$$
$$\langle 1, A\rangle \bullet \langle v, B\rangle \coloneqq \langle v, A \cup B\rangle$$
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$$\begin{aligned}
\langle 1, A\rangle \bullet \langle 1, B\rangle &:= \langle 1, A \cup B\rangle & \langle 1, A\rangle \parallel \langle 1, B\rangle &:= \langle 1, A \cup B\rangle \\
\langle u, A\rangle \bullet \langle 1, B\rangle &:= \langle u, A \cup B\rangle & \langle u, A\rangle \parallel \langle 1, B\rangle &:= \langle 1, A \cup B \cup \mathit{var}(u)\rangle \\
\langle 1, A\rangle \bullet \langle v, B\rangle &:= \langle v, A \cup B\rangle & \langle 1, A\rangle \parallel \langle v, B\rangle &:= \langle 1, A \cup B \cup \mathit{var}(v)\rangle \\
\langle u, A\rangle \bullet \langle v, B\rangle &:= \langle u \cdot v, A \cup B\rangle & \langle u, A\rangle \parallel \langle v, B\rangle &:= \langle u \cap v, A \cup B\rangle.
\end{aligned}$$

Translations to and from weak terms

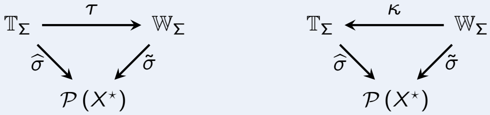
To weak terms

$$\begin{aligned} \tau(a) &:= \langle a, \emptyset \rangle & \qquad \tau(u \cdot v) &:= \tau(u) \bullet \tau(v) \\ \tau(1) &:= \langle 1, \emptyset \rangle & \qquad \tau(u \cap v) &:= \tau(u) \parallel \tau(v) \end{aligned}$$

From weak terms

$$\kappa(t, A) = \left(1 \cap \bigcap_{a \in A} a\right) \cdot t$$

Lemma



Correspondance

$$\forall\sigma,\ \widehat{\sigma}(u)=\widehat{\sigma}(v) \quad\Leftrightarrow\quad \tilde{\sigma}(\tau(u))=\tilde{\sigma}(\tau(v))$$

$$\Leftrightarrow\quad \tilde{\sigma}(\tau(u))\subseteq\tilde{\sigma}(\tau(v)) \text{ and } \tilde{\sigma}(\tau(u))\supseteq\tilde{\sigma}(\tau(v))$$

New problem

Given two weak terms v, w, is it the case that

$$\forall\sigma,\ \tilde{\sigma}(v)\subseteq\tilde{\sigma}(w)?$$

Containment of weak terms

$$\tilde{\sigma}\langle t, A\rangle = \begin{cases} \emptyset & \textit{if } \exists a \in A : \epsilon \notin \sigma(a) \\ \widehat{\sigma}(t) & \textit{otherwise} \end{cases}$$

Let $\langle u, A\rangle, \langle v, B\rangle$ be two weak terms.

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Let $\langle u, A\rangle, \langle v, B\rangle$ be two weak terms.

- if $\exists a \in B \setminus A$,
- if $B \subseteq A$,

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- if $\exists a \in B \setminus A$, there exists an alphabet X and an interpretation $\sigma : \Sigma \to X$ such that $\epsilon \notin \sigma(a)$ and $\tilde{\sigma}\langle u, A\rangle \neq \emptyset$, so:

$$\tilde{\sigma}\langle u, A\rangle \not\subseteq \tilde{\sigma}\langle v, B\rangle.$$

- if $B \subseteq A$,

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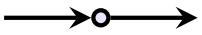
- if $B \subseteq A$, then $\forall\sigma,\ \tilde{\sigma}\langle u, A\rangle \subseteq \tilde{\sigma}\langle v, B\rangle$ is equivalent to:

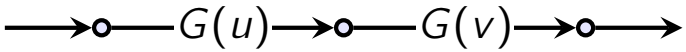
$$\forall\sigma, (\forall a \in A, \epsilon \in \sigma(a)) \Rightarrow \widehat{\sigma}(u) \subseteq \widehat{\sigma}(v).$$

Comparing series parallel terms

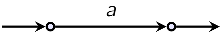
$\mathcal{G}(1) :=$



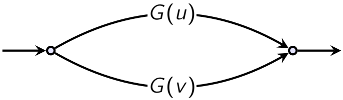
$\mathcal{G}(u \cdot v) :=$



$\mathcal{G}(a) :=$

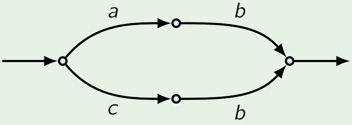


$\mathcal{G}(u \cap v) :=$

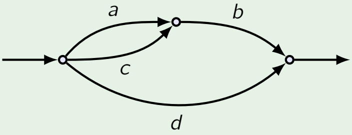


Example

$\mathcal{G}\left(((a \cap c) \cdot b) \cap d\right) =$



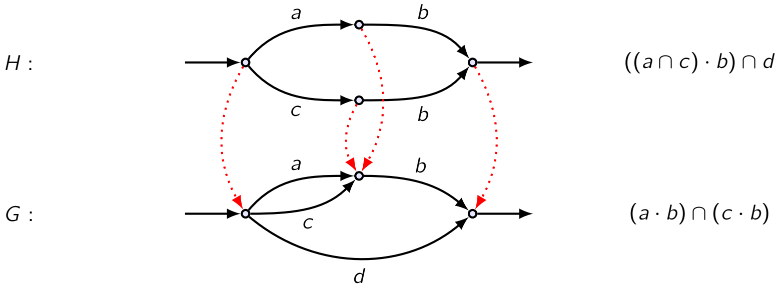
$\mathcal{G}\left((a \cdot b) \cap (c \cdot b)\right) =$



Preorder

Preorder on graphs

$G \blacktriangleleft H$ if there exists a graph morphism from H to G.



Characterization theorem

$$u, v \in \mathbb{SP}_\Sigma ::= a \mid u \cdot v \mid u \cap v$$

Theorem

$$\mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v)$$

Freyd & Scedrov, **Categories, Allegories**, *1990*

Andréka & Bredikhin, **The equational theory of union-free algebras of relations**, *1995*

Theorem

$$\forall u, v \in \mathbb{SP}_\Sigma,\ \mathrm{Rel} \models u \subseteq v \Leftrightarrow \mathrm{Lang} \models u \subseteq v.$$

Andréka, Mikulás & Németi, **The equational theory of Kleene lattices**, *TCS'11*

Preorder on weak graphs

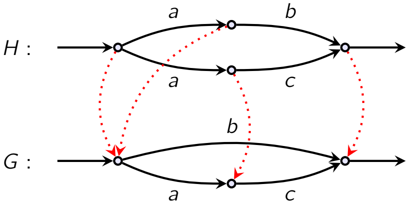
Definition

A weak graph is a pair of a graph and a set of tests.

Weak graph preorder

$\langle G, A\rangle \blacktriangleleft \langle H, B\rangle$ if $B \subseteq A$ and there is an A-weak morphism from H to G.

$A = \{a\}$



Characterisation Theorem

$\mathcal{G}\langle u, A\rangle := \langle \mathcal{G}(u), A\rangle$.

Lemma

Let $u \in \mathbb{W}_{\Sigma}$ be a weak term. There exists an alphabet X_u, a word $w_u \in X_u^\star$, and an interpretation σ_u such that:

$$\forall v \in \mathbb{W}_{\Sigma},\ w_u \in \tilde{\sigma_u}(v) \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v).$$

Theorem

$$(\forall \sigma,\ \tilde{\sigma}(u) \subseteq \tilde{\sigma}(v)) \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v).$$

Corollary

$\forall u, v \in \mathbb{T}_{\Sigma}$,

$$(\forall \sigma,\ \widehat{\sigma}(u) \subseteq \widehat{\sigma}(v)) \Leftrightarrow \mathcal{G}(\tau(u)) \blacktriangleleft \mathcal{G}(\tau(v)).$$

Axiomatisation of weak graph morphism

$$\frac{}{A \vdash_{sp} u \leq u} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} v \leq w}{A \vdash_{sp} u \leq w}$$

$$\frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cdot u' \leq v \cdot v'} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cap u' \leq v \cap v'}$$

$$\frac{}{A \vdash_{sp} u \cdot (v \cdot w) = (u \cdot v) \cdot w} \qquad \frac{}{A \vdash_{sp} u \cap (v \cap w) = (u \cap v) \cap w}$$

$$\frac{}{A \vdash_{sp} u \leq u \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq v \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq u}$$

$$\frac{var(u) \subseteq A}{A \vdash_{sp} v \leq u \cdot v} \qquad \frac{var(u) \subseteq A}{A \vdash_{sp} v \leq v \cdot u}$$

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$$\frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cdot u' \leq v \cdot v'} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} u' \leq v'}{A \vdash_{sp} u \cap u' \leq v \cap v'}$$

$$\frac{}{A \vdash_{sp} u \cdot (v \cdot w) = (u \cdot v) \cdot w} \qquad \frac{}{A \vdash_{sp} u \cap (v \cap w) = (u \cap v) \cap w}$$

$$\frac{}{A \vdash_{sp} u \leq u \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq v \cap u} \qquad \frac{}{A \vdash_{sp} u \cap v \leq u}$$

$$\frac{var(u) \subseteq A}{A \vdash_{sp} v \leq u \cdot v} \qquad \frac{var(u) \subseteq A}{A \vdash_{sp} v \leq v \cdot u}$$

Completeness lemma

$A \vdash_{sp} u \leq v$ if and only if there exists an A-weak morphism from $\mathcal{G}(v)$ to $\mathcal{G}(u)$.

Proof.

- if $A \vdash_{sp} u \leq v$ then there exists an A-weak morphism from $\mathcal{G}(v)$ to $\mathcal{G}(u)$: easy induction on the proof tree.

□

Proof.

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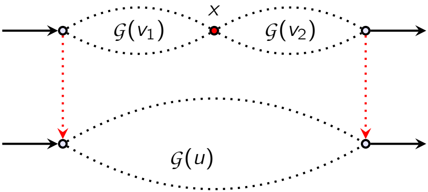

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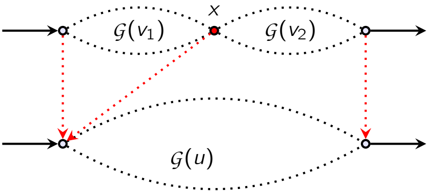

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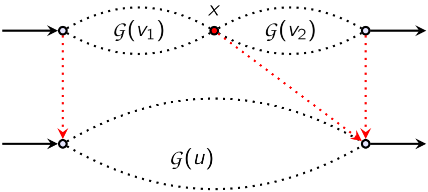

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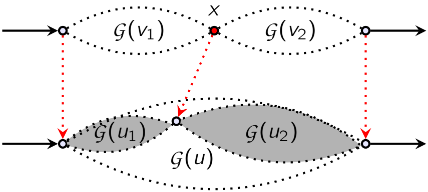
    - $\mathit{var}(v_2) \subseteq A$ and ϕ is an A-weak morphism from $\mathcal{G}(v_1)$ to $\mathcal{G}(u)$;
    - there are two terms u_1, u_2 such that $A \vdash_{sp} u \leq u_1 \cdot u_2$ and we can decompose ϕ into ϕ_1, ϕ_2 A-weak morphisms from $\mathcal{G}(v_1)$ and $\mathcal{G}(v_2)$ to $\mathcal{G}(u_1)$ and $\mathcal{G}(u_2)$.

□



Preorder on weak terms

Definition

$\vdash_w \langle u, A\rangle \leq \langle v, B\rangle$ if $B \subseteq A$ and

- either $u = 1$ and $\mathit{var}(v) \subseteq A$
- or $u, v \neq 1$ and $A \vdash_{sp} u \leq v$.

Lemma

$\vdash_w \langle u, A\rangle \leq \langle v, B\rangle$ if and only if $\mathcal{G}\langle u, A\rangle \blacktriangleleft \mathcal{G}\langle v, B\rangle$.

Preorder on terms

$$e, f \in \mathbb{T}_\Sigma ::= 1 \mid a \mid e \cdot f \mid e \cap f$$

$\vdash_t e = f$

$\langle \cdot, 1 \rangle$ is monoid; $\cap$ is associative, commutative and idempotent.

$$\begin{array}{lrcl}
\vdash_t & e \cap f \cdot g \cap h & = & (e \cap f \cdot g \cap h) \cap e \cdot f \\
\vdash_t & 1 \cap (e \cdot f) & = & 1 \cap (e \cap f) \\
\vdash_t & (1 \cap e) \cdot f & = & f \cdot (1 \cap e) \\
\vdash_t & ((1 \cap e) \cdot f) \cap g & = & (1 \cap e) \cdot (f \cap g)
\end{array}$$

Lemma

$$\vdash_t e = f \Rightarrow \vdash_w \tau(e) = \tau(f) \qquad \vdash_w u = v \Rightarrow \vdash_t \kappa(u) = \kappa(v).$$

$$\vdash_t e = \kappa(\tau(e)) \qquad \vdash_w u = \tau(\kappa(u)).$$

Lemma

$$\vdash_t e = f \Leftrightarrow \vdash_w \tau(e) = \tau(f)$$

Summing up

Theorem

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$$\begin{aligned}\vdash_t e = f &\Leftrightarrow \vdash_w \tau(e) = \tau(f)\\ &\Leftrightarrow \mathcal{G}(\tau(e)) \mathrel{\substack{\blacktriangleleft\\\blacktriangleright}} \mathcal{G}(\tau(f))\end{aligned}$$

Summing up

Theorem

$$\begin{aligned}\vdash_t e = f &\Leftrightarrow \vdash_w \tau(e) = \tau(f)\\ &\Leftrightarrow \mathcal{G}(\tau(e)) \mathrel{\blacktriangleleft\!\blacktriangleright} \mathcal{G}(\tau(f))\\ &\Leftrightarrow \mathrm{Lang} \models e = f\end{aligned}$$

Outline

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Mirror image

$$e, f \in \mathbb{T}^{\smile}_{\Sigma} ::= 1 \mid a \mid e \cdot f \mid e \cap f \mid e^{\smile}$$

$$\eta : \mathbb{T}^{\smile}_{\Sigma} \to \mathbb{T}_{\Sigma \cup \Sigma'}$$

Mirror image

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Consider the term $(1 \cap a \cap b)^{\smile} \cdot \left(b^{\smile} \cdot c\right)^{\smile}$.

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$$(1 \cap a^{\smile} \cap b^{\smile}) \cdot c^{\smile} \cdot b.$$

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2. double the variables under $1 \cap \_$:

$$\left(1 \cap a^{\smile} \cap a \cap b^{\smile} \cap b\right) \cdot c^{\smile} \cdot b.$$

Mirror image

$$e, f \in \mathbb{T}_{\Sigma}^{\smile} ::= 1 \mid a \mid e \cdot f \mid e \cap f \mid {\color{red} e^{\smile}}$$

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$$\left(1 \cap a^{\smile} \cap a \cap b^{\smile} \cap b\right) \cdot c^{\smile} \cdot b.$$

3. change the alphabet to $\Sigma \cup \Sigma'$:

$$(1 \cap a' \cap a \cap b' \cap b) \cdot c' \cdot b.$$

Mirror image

$$\vdash_c (a \cdot b)^{\smile} = b^{\smile} \cdot a^{\smile} \qquad \vdash_c (a \cap b)^{\smile} = a^{\smile} \cap b^{\smile} \qquad \vdash_c a^{\smile\smile} = a$$

$$\vdash_c 1^{\smile} = 1 \qquad \vdash_c 1 \cap a^{\smile} = 1 \cap a$$

Lemma

$$\vdash_c e = f \Leftrightarrow \vdash_t \eta(e) = \eta(f).$$

Lemma

$$\mathrm{Lang} \models e = f \Leftrightarrow \mathrm{Lang} \models \eta(e) = \eta(f).$$

Outline

Outline

Adding addition

$$e, f \in \mathrm{IMReg}^{-}\langle\Sigma\rangle ::= 1 \mid a \mid e \cdot f \mid e \cap f \mid e^{\smile} \mid e + f \mid 0.$$

Term language

$$\mathcal{L}(a) = \{a\} \qquad \mathcal{L}(1) = \{1\} \qquad \mathcal{L}(0) = \emptyset \qquad \mathcal{L}\left(e^{\smile}\right) = \left\{u^{\smile} \mid u \in \mathcal{L}(e)\right\}$$

$$\mathcal{L}(e+f) = \mathcal{L}(e) \cup \mathcal{L}(f) \qquad \mathcal{L}(e \cdot f) = \{u \cdot v \mid u \in \mathcal{L}(e)\ v \in \mathcal{L}(f)\}$$

$$\mathcal{L}(e \cap f) = \{u \cap v \mid u \in \mathcal{L}(e)\ v \in \mathcal{L}(f)\}$$

$$\vdash_a e + (f + g) = (e + f) + g \qquad \vdash_a e + f = f + e \qquad \vdash_a e + e = e$$

$$\vdash_a e + 0 = e \qquad \vdash_a e \cdot 0 = 0 \qquad \vdash_a 0 \cdot e = 0 \qquad \vdash_a e \cap 0 = 0$$

$$\vdash_a (e + f)^{\smile} = e^{\smile} + f^{\smile} \qquad \vdash_a 0^{\smile} = 0$$

Adding addition - axiomatisation

Lemma

$$\vdash_a e = \sum_{u \in \mathcal{L}(e)} u$$

Lemma

$$\vdash_a e \leq f \Rightarrow \forall u \in \mathcal{L}(e), \exists v \in \mathcal{L}(f) : \vdash_c u \leq v$$

Graph language

$$\mathcal{G}(e) = \{\mathcal{G} \circ \tau \circ \eta(u) \mid u \in \mathcal{L}(e)\}.$$

$$^{\blacktriangleleft}\mathcal{G}(e) = \{G \mid \exists H \in \mathcal{G}(e) : G \blacktriangleleft H\}$$

Theorem

$$\vdash_a e = f \Leftrightarrow {}^{\blacktriangleleft}\mathcal{G}(e) = {}^{\blacktriangleleft}\mathcal{G}(f)$$

Adding addition - completeness

$$\forall \sigma,\ \widehat{\sigma}(e) = \bigcup_{u \in \mathcal{L}(e)} \widehat{\sigma}(u).$$

Theorem

$$\mathrm{Lang} \models e = f \Leftrightarrow {}^{\blacktriangleleft}\mathcal{G}(e) = {}^{\blacktriangleleft}\mathcal{G}(f)$$

And now, for my next trick...

$$e, f \in \mathsf{IMReg}\langle\Sigma\rangle ::= 1 \mid a \mid e \cdot f \mid e \cap f \mid e^{\smile} \mid e + f \mid 0 \mid e^{\star}.$$

$$\mathcal{L}\left(e^{\star}\right) = \left\{u_1 \cdots u_n \mid n \in \mathbb{N} \text{ and } u_i \in \mathcal{L}\left(e\right)\right\}.$$

Theorem

$$\mathrm{Lang} \models e = f \Leftrightarrow {}^{\blacktriangleleft}\mathcal{G}\left(e\right) = {}^{\blacktriangleleft}\mathcal{G}\left(f\right)$$

That's all folks!

Thank you!

See more at:

`http://paul.brunet-zamansky.fr`

Outline